Wolverine Round

Names: ____________________

Team Name: ____________________

Instructions

1. Do not begin until instructed to by the proctor.
2. You will have 60 minutes to solve up to 24 problems, given in groups of 3.
3. Your score will be the number of correct answers. There is no penalty for guessing or incorrect answers.
4. **Only the official team answers will be graded.** If you are submitting the official answer sheet for your team, indicate this by writing "(OFFICIAL)" next to your team name. Do not submit any unofficial answer sheets.
5. No calculators or electronic devices are allowed.
6. All submitted work must be the work of your own team. You may collaborate with your team members, but no one else.
7. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
8. Do not discuss the problems with anyone outside of your team until all papers have been collected.
9. If you have a question or need to leave the room for any reason, please raise your hand quietly.
10. Good luck!

ACCEPTABLE ANSWERS

1. All answers must be simplified as much as reasonably possible. For example, acceptable answers include $\sin(1°)$, $\sqrt{43}$, or π^2. Unacceptable answers include $\sin(30°)$, $\sqrt{64}$, or 3^2.
2. All answers must be exact. For example, π is acceptable, but 3.14 or 22/7 is not.
3. All rational, non-integer numbers must be expressed in reduced form $\pm\frac{p}{q}$, where p and q are relatively prime positive integers and $q \neq 0$. For example, $\frac{2}{3}$ is acceptable, but $\frac{4}{6}$ is not.
4. All radicals must be fully reduced. For example, $\sqrt{24}$ is not acceptable, and should be written as $2\sqrt{6}$. Additionally, rational expressions cannot contain radicals in the denominator. For example, $\frac{1}{\sqrt{2}}$ is not acceptable, and should be written as $\frac{\sqrt{2}}{2}$.
5. Answers should be expressed in base 10 unless otherwise specified.
6. Complex numbers should be expressed in the form $a+bi$, where both a and b are written in a form compliant with the rules above. In particular, no complex denominators are allowed. For example, $\frac{1+2i}{1-2i}$ should be written as $-\frac{3}{5}+\frac{4}{5}i$ or $\frac{-3+4i}{5}$.
7. If a problem asks for all solutions, you may give the answers in any order. However, no credit will be given if any solution is missing or any solution is given but not correct.
8. Angle measurements should be given in radians unless otherwise specified.
9. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

Team Name: ______________________

1. (4 points) Let F, A, and H be distinct integers between 0 and 9, inclusive. Given that $F \cdot A \cdot H \cdot H \cdot H = 512$, find $F + A + H + H + H$.

1. 21

Solution: Note that $512 = 8^3$. However, we can not set $H = 8$, or else F and A would both be 1. So we set $H = 4$ and get $F \cdot A = 8$. Neither F nor A can equal 4 now, so one must equal 8 and the other 1. The answer is $8 + 1 + 4 + 4 + 4 = \boxed{21}$.

2. (4 points) The polynomial $x^2 - px + q$ has roots $p - 2026$ and $q - 2026$. Find $p + q$.

2. 6080

Solution: Using Vieta's formulas, we get that the sum of the roots is $p = p - 2026 + q - 2026$, meaning $q = 4052$. We also get that $q = (p-2026)(q-2026)$ and plugging in $q = 4052$ gives us $4052 = (p-2026) \cdot 2026$ so $p = 2028$. Our answer is $p + q = 2028 + 4052 = \boxed{6080}$.

3. (Game) On the Wolverine round, each set of three problems contains one estimation or game problem and two regular problems. Getting both regular problems correct on a set is called a *perfect packet*. The Wolverdjinn grants you one of two wishes: if you choose Wish A, you will receive 4 points for this problem. If you choose Wish B, you will receive 1 point for every perfect packet you submit during the entire Wolverine round. Your answer to this question should be one of the letters A or B, indicating the wish you choose.

3. **N/A**

Solution: 19 teams picked Wish A and received 4 points; 4 teams picked Wish B and received 3.5 points on average.

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4. (5 points) Let O denote the origin. Given a point P that is not O, define the *glow-up* of P as the unique point P' such that O, P, and P' are collinear, P and P' lie on the same side of O, and the length of line segment OP' is equal to the reciprocal of the length of line segment OP. Let L be the set of points (x, y) satisfying $6x + 8y \geq 9$. Let L' be the region defined as the set of points whose glow-up is in L. What is the area of L'?

4. $\frac{25\pi}{81}$

Solution: In essence, the glow-up point is an inversion with respect to the unit circle. Thus, any point (x, y) under inversion is $(X, Y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. Note that the glow-up of a glow-up maps to the original point, that is, the inversion map is its own inverse. Thus, we have the inverse mapping from inverted coordinates to the original coordinates $(x, y) = (\frac{X}{X^2+Y^2}, \frac{Y}{X^2+Y^2})$.

Plugging into the equation of L, L' is given by $(x - \frac{1}{3})^2 + (y - \frac{4}{9})^2 \leq \frac{25}{81}$, which is the equation of a disk with radius $\frac{5}{9}$.

Thus, the area is $\frac{25\pi}{81}$.

5. (5 points) What are all possible real numbers x such that $\cos^{-1}(\sin(x)) = 2x$?

5. $\frac{\pi}{6}$

Solution:

Taking cos of both sides, we have

$$\sin(x) = \cos(2x),$$

so

$$\sin(x) = 1 - 2\sin^2(x),$$

$$2\sin^2(x) + \sin(x) - 1 = 0,$$

$$\sin(x) = \frac{-1 \pm \sqrt{1 + 4 \cdot 2 \cdot 1}}{4} = \frac{1}{2}, -1.$$

Since the range of $\cos^{-1}$ is $[0, \pi]$, the only valid solution is $x = \boxed{\frac{\pi}{6}}$.

6. (Game) Let S denote the unit square with its bottom left corner at the origin. Submit an ordered pair (p_1, p_2) representing a point that is contained within S. Let R denote the set of points (r_1, r_2) in S such that for every point (q_1, q_2) submitted by other teams, the Euclidean distance between (r_1, r_2) and (p_1, p_2) is less than the Euclidean distance between (r_1, r_2) and (q_1, q_2). In other words, R is the set of points in S that have the property that your submitted point is the closest among all submitted points. The number of points you receive for this problem will be equal to 10 times the area of R.

6. **N/A**

Solution:

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7. (6 points) A *67-Haiku* is a poem consisting of only the 1-syllable word "Six" and the 2-syllable word "Seven". Furthermore, it must contain exactly the following components: a first line containing a 1-word title, a second line containing 6 syllables, a third line containing 7 syllables, and a fourth line containing 6 syllables. For example, a valid 67-Haiku could be:

Seven
Six Seven Seven Six
Seven Six Six Six Seven
Six Seven Six Seven

The total number of possible 67-Haikus can be expressed in the form $(6+7)^a(6\cdot 7)^b$ for non-negative integers a and b. What is $6(a+b)+7^a$?

7. 67

Solution: Let us count the number of possible choices for each line. For line 1, there are 2 choices: "Six" and "Seven". For lines 2, 3, and 4, we see that we can count them using a recurrence. Let f_n equal the number of possible lines with n syllables. We have $f_1 = 1$ and $f_2 = 2$. For $n \geq 3$, we see that if a line begins with "Six", then there are $n-1$ syllables left, which can be filled in f_{n-1} ways. If a line begins with "Seven", then there are $n-2$ syllables left, which can be filled in f_{n-2} ways. This results in the recurrence $f_n = f_{n-1} + f_{n-2}$. Using this recurrence, we compute $f_3 = 3, f_4 = 5, f_5 = 8, f_6 = 13$, and $f_7 = 21$. Therefore, the total number of possible 67-Haikus is $2f_6f_7f_6 = 2\cdot 13\cdot 21\cdot 13 = (6+7)^2(6\cdot 7)^1$, so $a = 2$ and $b = 1$. Then, our final answer is $6(2+1)+7^2 = \boxed{67}$.

8. (6 points) Let f be a cubic polynomial with integer coefficients. Given that $f(4) = 16$, $f(5) = 25$, $f(6) = 36$, and $f(7)$ is a square number greater than 49, what is the smallest possible value of $f(8)$?

8. 352

Solution: The polynomial is of the form $C(x-4)(x-5)(x-6)+x^2$ for some integer C. Then $f(7) = 6C + 49$ and we know this is a square number. Since all odd squares are equivalent to 1 mod 8, we can test multiples of 4 for C. After trying $C = 12$ we get that the smallest square is $f(7) = 121$, so $f(8) = 12\cdot 4\cdot 3\cdot 2 + 8^2 = \boxed{352}$.

9. (Game) Pick a real number from the interval $[1, 10]$; this will be your submission to this question, say x. If x ranks in the top 15% of all submissions when arranged in descending order, you will NOT score any points on this question. Else, you will score x points. Any ties crossing the 15% threshold will be counted as above the threshold; for example, if all the submissions were 10, none of you would score points.

9. **N/A**

Solution: The average submission was 5.46, and the Top 15% threshold was 8.5.

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10. (7 points) A function $f(x)$ is called *sus* if it satisfies $f(2)=2$ and $2f(x)=f(2x)(f(x)+2)$ for all $x \geq 2$. For what value of a is it true that every sus function $f(x)$ satisfies $f(a)=\frac{1}{5}$.

10. 1024

Solution: Make the substitution $g(x)=\frac{2}{f(x)}$.
We have $g(2)=1$ and $\frac{4}{g(x)}=\frac{2}{g(2x)}\left(\frac{2}{g(x)}+2\right) \implies g(2x)=1+g(x)$.
$f(a)=\frac{1}{5}$ is equivalent to $g(a)=10$.
Note that $g(2x)=1+g(x)$ implies that $g(x)$ increases by 1 every time the input doubles. Then, since $g(2)=1$, we have $g(2*2^9)=1+9$, so $g(1024)=10$. To see that $a=1024$ is the unique solution, note that the conditions only give unique values to inputs that are powers of 2. Furthermore, $g(x)$ is actually just the strictly increasing function $\log_2(x)$ on these inputs, guaranteeing uniqueness. Therefore, the answer is $a=\boxed{10/24}$.

11. (7 points) Let $z_1=20+26i$ and $z_2=4+12i$, where i denotes the imaginary unit defined by $i^2=-1$. Find the real value of k such that the equation $|z-z_1|^2+|z-z_2|^2=k$ has exactly one complex solution for z.

11. **226**

Solution: For simplicity in algebra, let $z=x+iy$, $z_1=x_1+iy_1$, and $z_2=x_2+iy_2$. Then, the equation simplifies to:

$$(x-x_1)^2+(y-y_1)^2+(x-x_2)^2+(y-y_2)^2=k$$

$$2x^2-2x(x_1+x_2)+x_1^2+x_2^2+2y^2-2y(y_1+y_2)+y_1^2+y_2^2=k$$

After completing the square, the equation simplifies to

$$(x-\bar{x})^2+(y-\bar{y})^2=\frac{k}{2}-\frac{(x_1-x_2)^2+(y_1-y_2)^2}{4}$$

where $\bar{x}=\frac{x_1+x_2}{2}$, $\bar{y}=\frac{y_1+y_2}{2}$. This is a circle centered at the midpoint of z_1 and z_2; thus, for the equation to have a single solution, the radius must equal zero. Plugging in $x_1=20$, $x_2=4$, $y_1=26$, $y_2=12$, we get $\boxed{k=226}$.

12. (Game) Congratulations on reaching halfway through the Wolverine Round! As a reward, we are offering all teams bonus points. We have 3 pools of points: Pool A has 50 points, Pool B has 60 points, and Pool C has 75 points. Your response to this question must be one of A, B, or C, which indicates which pool of points you choose. Points will be distributed proportionally within each pool to the teams that picked that pool. For example, if 12 teams picked Pool B, each of those 12 teams would receive $60/12=5$ points. (Score contribution from this problem won't be shown until all teams have turned in this set).

12. **N/A**

Solution: 2 teams picked Pool A and received 25 points each, 6 teams picked Pool B and received 10 points each, and 15 teams picked Pool C and received 5 points each.

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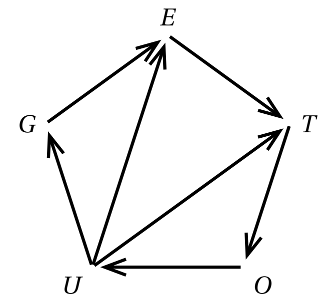
13. (8 points) Jacob wants to GET OUT. He is stuck in the pentagon shown below. Every minute, Jacob must walk to a new vertex. If Jacob is at vertex G, E, T, or O, he must follow the arrow in the diagram to the next vertex. However, if Jacob is at vertex U, there is an equal probability that he will walk to each of the vertices G, E, or T. If Jacob begins at vertex G at minute 0, what is the expected number of minutes that passes before he visits the vertices $GETOUT$ consecutively in that order?



13. 26

Solution: The first five vertices visited must be $GETOU$, so we will add 4 at the end to account for this. From vertex U, let E_G be the expected number of remaining vertices Jacob visits given that his last five vertices are $GETOU$, and let E_N be the expected number of remaining vertices he visits given his last five vertices are not $GETOU$. Then we can construct two equations:

$$E_G = \frac{1}{3}(5+E_G) + \frac{1}{3}(4+E_N) + \frac{1}{3}\cdot 1$$

$$E_N = \frac{1}{3}(5+E_G) + \frac{1}{3}(4+E_N) + \frac{1}{3}(3+E_N)$$

Solving these equations gives $E_G = 22$ and $E_N = 34$. So our answer is $E_G + 4 = \boxed{26}$.

14. (8 points) A mysterious village has 12 villagers. Its villagers have either red, green, or blue eyes. It has an interesting rule: if a villager learns of their own eye color on day n, they must leave the village on day $n+1$. Villagers cannot communicate with each other or directly observe their own eye color.

All villagers know that the only possible eye colors in their village are red, green, or blue, but do not know how many there are of each. On day 0, a traveler arrives in the village, and exclaims, "Wow, there are actually people with red eyes, people with green eyes, and people with blue eyes here!" In other words, there is at least one of each eye color. All of the villagers hear this, and know that all other villagers heard it as well.

Let R, G, B be the day in which the last red-eyed, green-eyed, and blue-eyed villagers leave the village, respectively. Every villager is a perfect logician, can always see the eye colors of all other villagers, and always knows when a villager leaves. If the village originally had 2 red-eyed villagers, 4 green-eyed villagers, and 6 blue-eyed villagers, what is the triplet (R, G, B)?

14. $(2,4,5)$

Solution: First, let the red-eyed, green-eyed, and blue-eyed villagers be $R_1, R_2, G_1, \ldots, G_4, B_1, \ldots, B_6$, respectively.

Let's first consider the situation from the red-eyed villagers perspective. Before day 0, both R_1 and R_2 see 1 other red-eyed villager. On day 1, neither villagers leave, because as far as they know, the other red-eyed villager could be the only red-eyed villager.

However, on day 1, R_1 sees that R_2 does not leave. This means that R_2 must be able to see another villager with red-eyes. R_1, not seeing any other villagers with red eyes, concludes that they themselves must have red eyes. This causes R_1 to leave the village on day 2. Similarly, R_2 also leaves the village on day 2.

Now, let's first consider a simplified case. If there had been only 3 green-eyed villagers, they would have all left the village on day 3. This is because in the case of 3 green-eyed villagers, each green-eyed villager can only see 2 other green-eyed villagers. If there were indeed only 2 green-eyed villagers, they would have already left on day 2. Them not leaving on day 2 means that there are more green-eyed villagers. Each green-eyed villager hence concludes that they themselves have green eyes on day 2, and hence leaves on day 3.

Using a similar logic, we know that all of the 4 green-eyed villagers will leave on day 4. This leaves behind only blue-eyed villagers. We can generalize the previous conclusion and realize that all villagers of the same colored eyes all leave on the same day. The blue-eyed villagers know this too, and the fact that they stayed means that they do not have red or green eyes. This means that the blue-eyed villagers learn that they have blue eyes on day 4, and hence all leave the village on day 5.

Hence, the answer is $\boxed{(2,4,5)}$.

You might be wondering if every villager knew that red, green, and blue-eyed villagers existed, what new information did the traveler bring in to change the balance? The answer is that the traveler informed everyone that everyone now knows that everyone now knows that everyone now knows ... that there are red, green, and blue eyes.

This is most easily understood from the perspective of R_1 and R_2. Before the traveler, R_1 can see that R_2 has red eyes, but does not know if R_2 knows if other villagers have red eyes. This is similar for the green-eyed and blue-eyed villagers, there are just more layers of knowing.

15. (Estimation)

Alice has accidentally mixed up two identical decks of cards! Each deck consists of 52 distinct cards. To separate the cards back into two separate decks, she goes through the 104 total cards one by one, and the first time she encounters a card she puts it in pile 1, and the second time she encounters it she puts it in pile 2.

She has perfect memory and keeps track of all the cards she has sorted in her head. She stops going through the pile when she is confident that she can just put the remaining cards in one of the piles and have the separation complete. Assuming that the 104 cards have an equal chance of starting with any order, what is the expected number of cards Alice goes through before she stops?

If the actual answer is A and your answer is X, the number of points you will receive for this question is $\max\left(0,\, 8 - \frac{|X-A|}{2}\right)$.

15. **92.1878926403**

Solution:

Let's consider when Alice would stop going through the cards. She stops exactly after going through at least one of each distinct type of cards. Afterwards, she knows she can just put the remaining cards in pile 2.

First, notice that, after going through 103 cards, Alice is guaranteed to have seen each type of card at least once.

Now, when would she stop after going through at most 102 cards? This happens exactly when the last 2 cards are of distinct types. In general, she would go through at most n cards precisely when the last $104-n$ cards are all distinct.

The probability Alice stops after going through at most 103 cards is 1. The probability for at most 102 cards is $\frac{102}{103}$, the probability for at most 101 cards is $\frac{100}{102}\cdot\frac{102}{103}$, the probability for at most 101 cards is $\frac{98}{101}\cdot\frac{100}{102}\cdot\frac{102}{103},\ldots$. The earliest Alice can possibly stop is after at most 52 cards, with probability $\frac{2}{53}\cdot\frac{4}{54}\cdot\ldots$.

Hence, the probability of Alice stopping after going through

Hence, our expected value is

$$E=\sum_{n=52}^{103} n\left(\prod_{i=0}^{103-n}\frac{104-2i}{104-i}-\prod_{i=0}^{104-n}\frac{104-2i}{104-i}\right)=92.1878926403.$$

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16. (9 points) Let r, s, t be the three complex roots of the equation $ax^3 + 2026x^2 + 2026x + b = 0$, where $a \neq 0$. How many ordered pairs of integers (a, b) are there such that the quantity $\frac{s+t}{r} + \frac{r+t}{s} + \frac{r+s}{t}$ is an integer?

16. 144

Solution: We note $\frac{s+t}{r} + \frac{r+t}{s} + \frac{r+s}{t} = \frac{(r+s+t)(rs+st+rt)}{rst} - 3 = \frac{2026^2}{ab} - 3$ by Vieta's formulas. Then, it suffices to count the number of (a, b) such that ab divides $2026^2 = 2^2 \cdot 1013^2$. For each prime, we distribute its exponent among a, b, and the unused remainder using stars and bars to get $\binom{2+3-1}{3-1}\binom{2+3-1}{3-1} = 36$ ordered pairs of positive integers. Since a and/or b can be negative, our final answer is $2 \cdot 2 \cdot 36 = \boxed{144}$.

17. (9 points) Alice and Bob are playing a game with infinite rounds, starting at round 1. In round n, Alice and Bob each flip identical independent coins that show heads with probability p_n, where p_n satisfies $p_n(1-p_n) = \frac{5}{17 \cdot 3^n + 15}$. In each round, if one player's coin shows heads and the other shows tails, the player with heads wins. Otherwise, they continue to the next round. What is the probability that Alice eventually wins?

17. $\frac{5}{44}$

Solution: First, the probability that the two coins agree on round n is

$$q_n = p_n \cdot p_n + (1-p_n) \cdot (1-p_n) = 2p_n^2 - 2p_n + 1 = 1 - 2p_n(1-p_n)$$

Plugging in our given expression,

$$q_n = 1 - \frac{2 \cdot 5}{17 \cdot 3^n + 15} = \frac{17 \cdot 3^n + 5}{17 \cdot 3^n + 15}.$$

Therefore, the probability that the game never ends is

$$P(\text{never ends}) = \prod_{n=1}^{\infty} q_n = \prod_{n=1}^{\infty} \frac{17 \cdot 3^n + 5}{17 \cdot 3^n + 15} = \prod_{n=1}^{\infty} \frac{17 + 5 \cdot 3^{-n}}{17 + 5 \cdot 3^{-(n-1)}}$$

We can see that this sum telescopes, hence

$$P(\text{never ends}) = \frac{17 + 5 \cdot 3^{-\infty}}{17 + 5 \cdot 3^{1-1}} = \frac{17}{22}$$

After accounting for that, by symmetry, Alice and Bob have the same chance of eventually winning, hence

$$P(\text{Alice eventually wins}) = \frac{1}{2}(1 - P(\text{never ends})) = \frac{1}{2}(1 - \frac{17}{22}) = \boxed{\frac{5}{44}}.$$

18. (Estimation) Gauss loves playing with numbers! One day, he decides to play the following game:

He starts with 1, and during each step, he decides to multiply the current number by either 2 or 3, with equal chance. If he does this 100 times, what's the expected number of digits of the resulting number? If the correct answer is D and your response is X, your score for this question will be $\frac{9}{(X-D)^2+1}$.

18. 39.4139080996

Solution: 2^{100} has 31 digits, and 3^{100} has 48 digits, so the answer should be somewhere in that range.

Although there are 2^{100} possible sequences of choices, the resulting number only depends on how many times Gauss chooses 3. Let K be that number. Then $K \sim \text{Binomial}(100, \frac{1}{2})$, and the final number is

$$N = 2^{100-K}3^K = 2^{100}\left(\frac{3}{2}\right)^K.$$

Hence the number of digits is

$$D = \lfloor \log_{10}(N) \rfloor + 1 = \left\lfloor 100\log_{10} 2 + K\log_{10}\left(\frac{3}{2}\right)\right\rfloor + 1.$$

Now $100\log_{10} 2 \approx 30.103$ and $\log_{10}(\frac{3}{2}) \approx 0.17609$, so D ranges from 31 to 48. More specifically, $D = 31$ for $0 \le K \le 5$, $D = 32$ for $6 \le K \le 10$, and so on, up to $D = 48$ for $96 \le K \le 100$. Therefore,

$$\mathbb{E}[D] = \sum_{k=0}^{100}\left(\left\lfloor 100\log_{10} 2 + k\log_{10}\left(\frac{3}{2}\right)\right\rfloor + 1\right)\binom{100}{k}\frac{1}{2^{100}}.$$

Evaluating this sum gives

$$\mathbb{E}[D] \approx 39.4139080996.$$

So the expected number of digits is about $\boxed{39.414}$.

Team Name: ____________________

19. (10 points) Let $\theta = \frac{\pi}{60}$, and define the sequence v_n where $v_1 = 0$ and $v_n = v_{n-1} + \tan(\theta)\tan((n-1)\theta)\tan(n\theta)$. What is v_{75}? You may express your answer in terms of $\tan(\frac{\pi}{60})$.

19. $1 - 75\tan(\frac{\pi}{60})$

It was pointed out during the competition that this problem is actually incorrect, because $\tan(30 \cdot \frac{\pi}{60}) = \tan(\frac{\pi}{2})$ **is undefined, causing** v_{30} **onwards to be undefined. Hence, this problem did not contribute to total scores for any team.**

Solution: We know the tangent formula $\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$, moving around, we get the formula $\tan(a)\tan(b) = 1 - \frac{\tan(a)+\tan(b)}{\tan(a+b)}$.

Now, plugging this formula into the above, we get $v_n = v_{n-1} + (1 - \frac{\tan(\theta)+\tan((n-1)\theta)}{\tan(n\theta)}) \cdot \tan(n\theta) = v_{n-1} + \tan(n\theta) - \tan((n-1)\theta) - \tan(\theta)$.

We can now telescope, so $v_n = v_n - v_1 = \sum_{i=1}^{n-1} v_{i+1} - v_i = \sum_{i=1}^{n-1} \tan((i+1)\theta) - \tan(i\theta) - \tan(\theta) = \tan(n\theta) - \tan(\theta) - (n-1)\tan(\theta) = \tan(n\theta) - n\tan(\theta)$.

Plugging in 75, we get $v_{75} = \tan(75 \cdot \frac{\pi}{60}) - 75\tan(\theta) = \tan(\frac{5\pi}{4}) - 75\tan(\theta) = 1 - 75\tan(\theta) = 1 - 75\tan(\frac{\pi}{60})$.

20. (10 points) Let x be chosen uniformly at random from the interval $(0, 2026]$. What is the probability that $\lceil \log_2 x \rceil = \log_2 \lceil x \rceil$?

20. $\frac{21}{4052}$

Solution: Case 1: $0 < x \le 1$
Since $0 < x \le 1$, $\log_2 \lceil x \rceil = \log_2 1 = 0$.
Then, the condition becomes $\lceil \log_2 x \rceil = 0$. This occurs when $-1 < \log_2 x \le 0 \iff \frac{1}{2} < x \le 1$. Therefore, when $0 < x \le 1$, the interval that satisfies the condition has length $\frac{1}{2}$.
Case 2: $1 < x \le 2026$
Let us rearrange the condition equation:
$\lceil \log_2 x \rceil = \log_2 \lceil x \rceil$
$\iff 2^{\lceil \log_2 x \rceil} = \lceil x \rceil$
$\iff 2^{\lceil \log_2 x \rceil} - 1 < x \le 2^{\lceil \log_2 x \rceil}$
The right inequality always holds since $\log_2 x \le \lceil \log_2 x \rceil$.
It remains to find when $2^{\lceil \log_2 x \rceil} - 1 < x$
$\iff 2^{\lceil \log_2 x \rceil} < x + 1$
$\iff \lceil \log_2 x \rceil < \log_2(x+1)$
Suppose $n - 1 < \log_2 x \le n \iff 2^{n-1} < x \le 2^n$ for some integer n.
Then, $\lceil \log_2 x \rceil = n$, so the condition becomes $n < \log_2(x+1)$
$\iff 2^n < x + 1$
$\iff 2^n - 1 < x$.
Therefore, the condition is true exactly when there exists an integer n such that both $2^{n-1} < x \le 2^n$ and

$2^n - 1 < x$ are true. This becomes $2^n - 1 < x \leq 2^n$ since $x > 1 \implies \log_2 x > 0 \implies n = \lceil \log_2 x \rceil \geq 1 \implies 2^n - 1 \geq 2^{n-1}$.
Therefore, we must sum the lengths of all sub-intervals of the form $(2^n - 1, 2^n]$ within the interval $(1, 2026]$. There are exactly 10 of these, each of length 1, so in case 2, the total length of intervals that satisfy the condition is 10.
Then, the final answer is $\frac{\frac{1}{2}+10}{2026} = \boxed{\frac{21}{4052}}$

21. (Estimation) Define the rank of the word as its relative position in the list of all permutations of its letters, with the permutations arranged in alphabetical order. For example, the word "BAC" would have rank(BAC) = 3, because it appears third in the list (ABC, ACB, BAC, BCA, CAB, CBA). Consider the word MATH. A random letter from the alphabet is chosen and appended to MATH to form a new word. Estimate the expected rank of this new word. If the actual expected rank is R and your answer is X, the number of points you will receive for this question is $\frac{30-|R-X|}{3}$.

21. **R = 59.07**

Solution:

Team Name: ________________________

22. (11 points) What is the minimum positive integer n such that there exist **distinct** primes p and q such that n^3+27 is divisible by p^3, and n^2+9 is divisible by q^2.

22. 21

Solution: We seek the minimum positive integer n such that there exist **distinct** primes p and q satisfying

$$p^3 \mid n^3+27 \qquad \text{and} \qquad q^2 \mid n^2+9.$$

We will split into cases depending on whether $p=3$.

First, note that

$$n^3+27=(n+3)(n^2-3n+9).$$

Also,

$$\gcd(n+3,n^2-3n+9)=\gcd(n+3,27),$$

so the two factors can only share powers of 3. Therefore, for any prime $p \neq 3$, the divisibility $p^3 \mid n^3+27$ implies that p^3 divides exactly one of $n+3$ or n^2-3n+9.

Case 1: $p=3$**.**

Then $27 \mid n^3+27$, so $3 \mid n$. Write $n=3m$. The second condition becomes

$$q^2 \mid n^2+9=9(m^2+1).$$

Since $q \neq p=3$, we must have

$$q^2 \mid m^2+1.$$

First, $q=2$ is impossible because no square is $\equiv 3 \mod 4$. Also, if an odd prime q divides m^2+1, then $m^2 \equiv -1 \pmod{q}$, so -1 is a quadratic residue modulo q. By Euler's criterion, this happens if and only if

$$(-1)^{(q-1)/2}=1,$$

which is equivalent to $q \equiv 1 \pmod{4}$. Thus the smallest possible odd prime is $q=5$, and trying this, we have

$$m^2 \equiv -1 \pmod{25}.$$

and the smallest positive solution is $m=7$, because $7^2=49 \equiv -1 \pmod{25}$. Hence the smallest n for $q=5$ is

$$n=3m=21.$$

If we had $q>5$, since $q \equiv 1 \mod 4$, we would have $q \geq 13$. But this would force $m^2+1 \geq 13^2=169$, or $m \geq 12$, and $n \geq 36$, so 21 is the minimum possible value for this case.

Case 2: $p \neq 3$**.**

We now show that no $n<21$ works in this case. Again. $q=2$ is impossible because no square is $\equiv 3 \mod 4$. If $q=3$, then $9 \mid n^2+9$, so $3 \mid n$. Write $n=3m$. Since $n<21$, we have $m \leq 6$. Then

$$n^3+27=27(m^3+1)=27(m+1)(m^2-m+1).$$

Now we need p^3 dividing $(m+1)(m^2-m+1)$. But for $m \le 6$, we have $m+1 \le 7$, so $m+1$ contains no prime cube, and

$$m^2 - m + 1 \in \{1, 3, 7, 13, 21, 31\},$$

none of which contains a prime cube either. So no such $p \neq 3$ exists.

Thus, for $n < 21$, we must have $q \neq 3$. Since $q^2 \mid n^2 + 9$ and $n < 21$, we have

$$n^2 + 9 < 21^2 + 9 = 450.$$

Also, since $q \neq 3$ and $q \mid n^2 + 9$, we have $n^2 \equiv -9 \pmod{q}$. Because 3 is invertible modulo q, this implies $(n \cdot 3^{-1})^2 \equiv -1 \pmod{q}$, so -1 is a quadratic residue modulo q. Hence $q \equiv 1 \pmod{4}$, and the only possibilities for q are $5, 13, 17, \ldots$. But if $q \ge 13$, then $q^2 \ge 169$ and $3q^2 > 450$, so the only possibilities are

$$n^2 + 9 = q^2 \quad \text{or} \quad n^2 + 9 = 2q^2.$$

Checking $q = 13$ and $q = 17$ gives

$$n^2 = 160, 329, 280, 569,$$

none of which is a square. Hence $q = 5$ is the only possibility for $n < 21$.

So we must have

$$n^2 \equiv -9 \equiv 16 \pmod{25},$$

which gives

$$n \equiv \pm 4 \pmod{25}.$$

Among positive integers less than 21, this forces $n = 4$. But then

$$4^3 + 27 = 91,$$

which is not divisible by any prime cube. So this case is impossible for $n < 21$.

Therefore the minimum possible value is

$$\boxed{21}.$$

23. (11 points) Let $f(x) = a_{2026}x^{2026} + a_{2025}x^{2025} + \ldots a_0$, and $g(x) = b_{2026}x^{2026} + b_{2025}x^{2025} + \ldots b_0$, where Each a_i and b_i for $0 \le i \le 2026$ is chosen independently and uniformly from $\{1, -1\}$. The expected number of positive coefficients of $f(x) \cdot g(x)$ can be written in the form of $\frac{a}{p} - \frac{b}{q^c}\binom{2026}{1013}$, where a, b, c, p, q are positive integers, a, p are relatively prime, b, q are relatively prime, and p, q are prime. What is $a + b + c + p + q$?

23. **8112**

Solution: Let $h(x) = f(x) \cdot g(x) = c_{4052}x^{4052} + c_{4051}x^{4051} + \ldots + c_0$, then we know

$$\begin{aligned}
c_{4052} &= a_{2026}b_{2026} \\
c_{4051} &= a_{2026}b_{2025} + a_{2025}b_{2026} \\
&\ldots \\
c_{2027} &= a_{2026}b_1 + a_{2025}b_2 + \ldots + a_1 b_{2026} \\
c_{2026} &= a_{2026}b_0 + a_{2025}b_1 + \ldots + a_0 b_{2026} \\
c_{2025} &= a_{2025}b_0 + a_{2024}b_1 + \ldots + a_0 b_{2025} \\
&\ldots \\
c_1 &= a_1 b_0 + a_0 b_1 \\
c_0 &= a_0 b_0
\end{aligned}$$

Now, by linearity of expectation,

$$\mathbb{E}(\text{Number of positive coefficients}) = P(c_{4052} > 0) + P(c_{4051} > 0) + \ldots + P(c_0 > 0).$$

Notice that every c_k is composed of the sum of $a_i b_j$ terms. Each term also has a equal chance of being $+1$ or -1. Also, notice that if k is even, then c_k is the sum of an odd number of terms which are either 1 or -1, so c_k cannot be 0. This means that by symmetry, $P(c_k > 0) = \frac{1}{2}$ for even k.

Now, let's consider if k is odd. By symmetry, $P(c_k > 0) = \frac{1}{2}(1 - P(c_k = 0))$. Now, if c_k is the sum of $2m$ terms, we have

$$P(c_k = 0) = \frac{\binom{2m}{m}}{2^{2m}} = \frac{\binom{2m}{m}}{4^m}$$

since we need exactly m of the $a_i b_j$ to be $+1$ and the other m to be -1 in order for the c_k to be 0.

A well known formula is the following:

$$\binom{-\frac{1}{2}}{m} = \frac{(-1)^m}{4^m}\binom{2m}{m}$$

because we can evaluate

$$\begin{aligned}
\binom{-\frac{1}{2}}{m} &= \frac{(-\frac{1}{2})(-\frac{1}{2} - 1)\ldots(-\frac{1}{2} - m + 1)}{m!} \\
&= \frac{(-1)^m}{2^m} \frac{1 \cdot 3 \cdot \ldots (2m-1)}{m!} \\
&= \frac{(-1)^m}{2^m} \frac{(2m)!}{2 \cdot 4 \cdot \ldots 2m \cdot m!} \\
&= \frac{(-1)^m}{2^m 2^m} \frac{(2m)!}{m! m!} \\
&= \frac{(-1)^m}{4^m} \binom{2m}{m}.
\end{aligned}$$

Now, plugging this into our formula above, we have for k is odd,

$$P(c_k = 0) = (-1)^m \binom{-\frac{1}{2}}{m}.$$

Now, notice that for odd k, the number of $a_i b_j$ in the expression for c_k is $k+1$ for $k < 2026$ and $4053 - k$ for $k > 2026$ (so 2 terms for c_1, 4 for c_3, $\ldots$, 2026 for c_{2025} and $c_{2027}, \ldots 2$ for c_{4051}), hence

$$\begin{aligned}
\mathbb{E}(\text{Number of positive coefficients}) &= P(c_{4052} > 0) + P(c_{4051} > 0) + \ldots + P(c_0 > 0) \\
&= 2027 \cdot \frac{1}{2} + 2\sum_{m=1}^{1013} \frac{1}{2} \cdot \left(1 - (-1)^m \binom{-\frac{1}{2}}{m}\right)
\end{aligned}$$

2027 $\sum^{1013}$

24. (Estimation) Drew the Dryptosaurus draws a circle of radius 2026. Then, he draws a second circle of radius 2025 centered at a point chosen uniformly at random from the interior of the first circle. He continues this process, drawing a circle of radius one less than the radius of the previous circle and centered at a point chosen uniformly at random from the interior of the previous circle, until he finally draws a circle of radius 1. Estimate the expected distance between the centers of the first and last circles. If the true expected distance is x, and your estimate is y, then the number of points you will receive for this problem is given by the following formula:

$$11 \cdot \min\left(\frac{x}{y}, \frac{y}{x}\right).$$

24. 33008.34781121476

Solution: $\boxed{33008.34781121476}$ (approximated using 10,000,000 trials with a 95% confidence interval of 33008.35 ± 10.69)

A crude approach to estimate this without using advanced machinery is as follows:

Let x_n be the center of the n'th circle, which has radius $r_n = 2027 - n$.

First, we estimate the expected distance of x_{n+1} from x_n. With some basic calculus, this can be shown to be equal to $\frac{2}{3}r_n$. However, let us avoid calculus, and crudely approximate this by finding the radius of a circle with half the area of the n'th circle.

$\pi r^2 = \frac{1}{2}\pi r_n^2 \implies r = \frac{\sqrt{2}}{2} r_n$

Next, as a rough model, we replace the random process by assuming that x_{n+1} is chosen such that line $x_n x_{n+1}$ is perpendicular to line $x_1 x_n$, since the other countributions roughly cancel out. Then, the expected distance becomes:

$$\begin{aligned}
&\sqrt{\sum_{n=1}^{2026} \left(\frac{\sqrt{2}}{2} r_n\right)^2} \\
&= \frac{\sqrt{2}}{2}\sqrt{\sum_{n=1}^{2026} r_n^2} \\
&= \frac{\sqrt{2}}{2}\sqrt{\sum_{n=2}^{2026} n^2} \\
&\approx \frac{\sqrt{2}}{2}\sqrt{\frac{2026(2026+1)(2(2026)+1)}{6}} \\
&\approx \frac{\sqrt{2}}{2}\sqrt{\frac{2026^3}{3}} \\
&\approx \frac{\sqrt{2}}{2}\frac{2026(45)}{\sqrt{3}} \\
&= \sqrt{6}(1013)(15) \\
&\approx (2.4)(1000)(15) \\
&= 36000
\end{aligned}$$

A more precise approach is to use the 2D central limit theorem, which gives an estimate of:

$$\sqrt{\frac{\pi}{8}\sum_{n=2}^{2026} n} \approx 33006$$