TEAM ROUND

Names: ______________________________

Team Name: ______________________________

INSTRUCTIONS

1. Do not begin until instructed to by the proctor.
2. You will have 60 minutes to solve 10 problems. Your score will be the number of correct answers. There is no penalty for guessing or incorrect answers.
3. **Only the official team answers will be graded.** If you are submitting the official answer sheet for your team, indicate this by writing "(OFFICIAL)" next to your team name. Do not submit any unofficial answer sheets.
4. No calculators or electronic devices are allowed.
5. All submitted work must be the work of your own team. You may collaborate with your team members, but no one else.
6. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
7. Do not discuss problems with anyone outside of your team until all papers have been collected.
8. If you have a question or need to leave the room for any reason, please raise your hand quietly.

ACCEPTABLE ANSWERS

1. All answers must be simplified as much as reasonably possible. For example, acceptable answers include $\sin(1^\circ)$, $\sqrt{43}$, or π^2. Unacceptable answers include $\sin(30^\circ)$, $\sqrt{64}$, or 3^2.
2. All answers must be exact. For example, π is acceptable, but 3.14 or 22/7 is not.
3. All rational, non-integer numbers must be expressed in reduced form $\pm\frac{p}{q}$, where p and q are relatively prime positive integers and $q \neq 0$. For example, $\frac{2}{3}$ is acceptable, but $\frac{4}{6}$ is not.
4. All radicals must be fully reduced. For example, $\sqrt{24}$ is not acceptable, and should be written as $2\sqrt{6}$. Additionally, rational expressions cannot contain radicals in the denominator. For example, $\frac{1}{\sqrt{2}}$ is not acceptable, and should be written as $\frac{\sqrt{2}}{2}$.
5. Answers should be expressed in base 10 unless otherwise specified.
6. Complex numbers should be expressed in the form $a + bi$, where both a and b are written in a form compliant with the rules above. In particular, no complex denominators are allowed. For example, $\frac{1+2i}{1-2i}$ should be written as $-\frac{3}{5} + \frac{4}{5}i$ or $\frac{-3+4i}{5}$.
7. If a problem asks for all solutions, you may give the answers in any order. However, no credit will be given if any solution is missing or any solution is given but not correct.
8. Angle measurements should be given in radians unless otherwise specified.
9. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

TEAM ROUND

1. Cayley the Crow comes across a cylindrical container of radius 88 and height 96 that is initially filled halfway with water. To raise the water level to the top of the container, Cayley begins to add solid spherical stones into the container, first of radius 1, then radius 2, then radius 3, and so on, until she finally adds a stone of radius n, at which point she notices that the water perfectly fills the container. Assume the stones are all fully submerged in the water. Compute n.

1. **32**

Solution:

$$\begin{aligned}
\frac{4}{3}\pi 1^3 + \frac{4}{3}\pi 2^3 + \cdots + \frac{4}{3}\pi n^3 &= \frac{1}{2}\pi(96)(88^2)\\
\frac{4}{3}\pi(1^3 + 2^3 + \cdots + n^3) &= \frac{1}{2}\pi(96)(88^2)\\
1^3 + 2^3 + \cdots + n^3 &= (36)(88^2)\\
(1 + 2 + \cdots + n)^2 &= (36)(88^2)\\
1 + 2 + \cdots + n &= (6)(88)\\
\frac{n(n+1)}{2} &= (16)(33)\\
n(n+1) &= (32)(33)\\
n &= \boxed{32}
\end{aligned}$$

2. An integer n is called *puny* if, out of all of the possible distinct integers that can be formed by permuting the digits of n, at least half of the integers are **strictly** greater than n. For example, 1111 would not be puny, as there is only one permutation, itself, and it is not strictly greater than itself, but 1234 would be puny. How many 4-digit puny integers not containing the digit 0 are there?

2. 3276

Solution: We can group all 4-digit positive integers where none of the digits are 0 into groups where every integer in the same group have the same multi-set of digits. A number is therefore *puny* if any only if it is strictly in the first half of integers in its group.

Notice that unless every digit is the same, there is always an even number of integers in each group. If every digit is the same, there is only one number in the group, and it's not *puny*. Hence, our answer is $\frac{9 \cdot 9 \cdot 9 \cdot 9 - 9}{2} = \boxed{3276}$.

3. A *happy* sequence is a sequence of non-negative integers with the property that for every triplet of consecutive terms in the sequence, the sum of the first two terms is at least 3 greater than the third term. How many happy sequences of length 2026 are there such that the first two terms are both equal to 3?

3. 14

Solution: First, note that starting with 3, 3, choosing the maximal next term at each step results in the sequence 3, 3, 3, ..., 3. Therefore, the only terms that can appear in the sequence are 0, 1, 2, and 3. Furthermore, observe that the sequence must be non-increasing. Then, we may build each happy sequence starting at the first term in which the sequence begins to decrease. By exhaustion, we find that the valid ending terms for happy sequences are:

$$..., 3, 3, 3, 3, 3$$

$$..., 3, 3, 3, 3, 2$$

$$..., 3, 3, 3, 3, 1$$

$$..., 3, 3, 3, 3, 0$$

$$..., 3, 3, 3, 2, 2$$

$$..., 3, 3, 3, 2, 1$$

$$..., 3, 3, 3, 2, 0$$

$$..., 3, 3, 3, 1, 1$$

$$..., 3, 3, 3, 1, 0$$

$$..., 3, 3, 3, 0, 0$$

$$..., 3, 3, 2, 2, 1$$

$$..., 3, 3, 2, 2, 0$$

$$..., 3, 3, 2, 1, 0$$

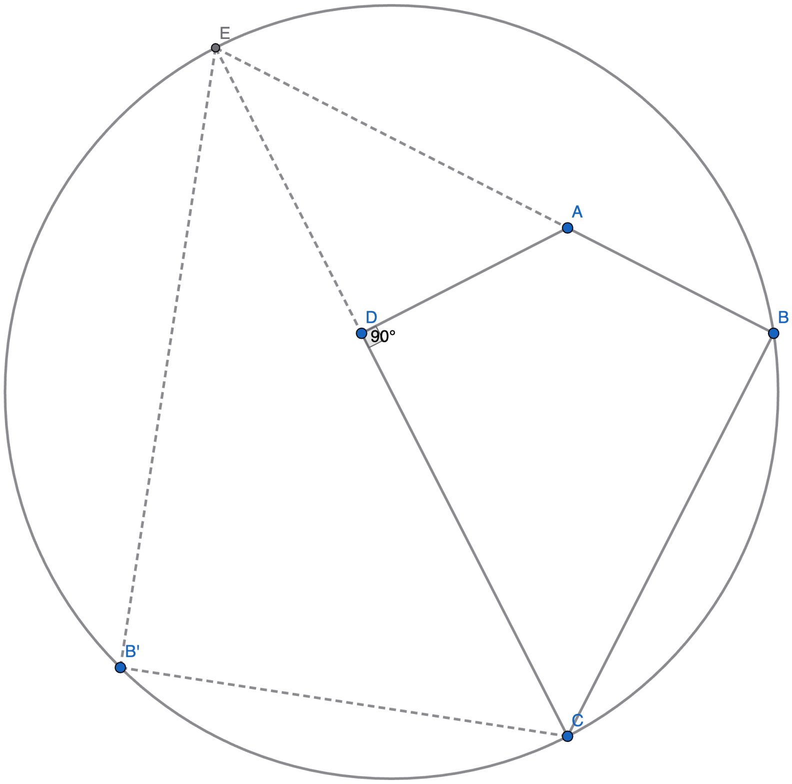
$$..., 3, 2, 2, 1, 0$$

Therefore, the answer is $\boxed{14}$.

4. In quadrilateral $ABCD$, $AB = AD = 2$, $CB = CD = 3$ and $\angle ADC = 90°$. Let B' be the reflection of B across $\overline{CD}$. What is the radius of the circumcircle of $\triangle BCB'$?

4. $\frac{39}{10}$

Solution: This is the intended solution.



Note that this diagram is not drawn to scale.

Construct a point E to be the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. Since $ABCD$ is a kite, $\angle ABC = \angle ADC = 90^\circ$. This means that $\overline{CE}$ is a diameter of the circumcircle of $\triangle CBE$.

Notice that since B' is the reflection of B over $\overline{CD}$ which is a diameter, it is also on the circumcircle of $\triangle CBE$. This means that the circumcircle of $\triangle BCB'$ is the same circle as the circumcircle of $\triangle CBE$, so we will aim to find the radius of the circumcircle of $\triangle CBE$.

Notice that $\triangle EDA \sim \triangle EBC$. Let $ED = x$, then $EA = \sqrt{x^2+4}$. Using $\frac{BC}{DA} = \frac{EB}{ED}$, we have

$\frac{3}{2} = \frac{\sqrt{x^2+4}+2}{x}$,

$3x - 4 = 2\sqrt{x^2+4}$,

$9x^2 - 24x + 16 = 4x^2 + 16$,

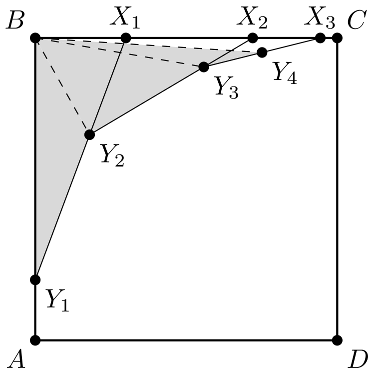
$5x^2 - 24x = 0$,

$x = \frac{24}{5}$, as we ignore the 0 solution.

This means that $EC = 3 + \frac{24}{5} = \frac{39}{5}$, therefore, our answer is $R = \frac{EC}{2} = \boxed{\frac{39}{10}}$.

5. Let $ABCD$ be a unit square. Let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ be real numbers chosen independently and uniformly at random from the interval $[0,1]$. Define X_1 as the unique point on line segment $\overline{BC}$ such that $CX_1 = x_1 \cdot CB$. In other words, the distance between C and X_1 is x_1 times the distance between C and B. Define Y_1 as the unique point on line segment $\overline{AB}$ such that $BY_1 = y_1 \cdot BA$. For all $i \geq 2$, define X_i as the unique point on line segment $\overline{X_{i-1}C}$ such that $CX_i = x_i \cdot CX_{i-1}$, and define Y_i as the unique point on line segment $\overline{Y_{i-1}X_{i-1}}$ such that $X_{i-1}Y_i = y_i \cdot X_{i-1}Y_{i-1}$. What is the expected sum of the areas of all triangles of the form $\triangle BY_iY_{i+1}$?

Shown below is an example where the first few X_i and Y_i are shown where $x_1 = 0.7, x_2 = 0.4, x_3 = 0.2$ and $y_1 = 0.8, y_2 = 0.4, y_3 = 0.3, y_4 = 0.5$. The first three triangles of the form $\triangle BY_iY_{i+1}$ have also been shaded in gray.



5. $\frac{1}{6}$

Solution: Let S be the union of the triangles.

Instead of thinking of S as the union of triangles of the form $\triangle BY_iY_{i+1}$ for $i \geq 1$, it will be easier to treat S as the union of triangles of the form $\triangle X_iX_{i+1}Y_{i+1}$ where $i \geq 0$ and we define X_0 as B for convenience. Note that the interiors of each of these triangles are pairwise-disjoint so we can consider their areas separately.

Recall that the area of the triangle is given by $\frac{1}{2}bh$ where b is the base length and h is the length of a perpendicular from the base side to the opposite vertex. We can therefore think about the points $X_1, X_2, \ldots$ and $Y_2, Y_3, \ldots$ as a scaled down version of the original problem by decrementing all of the indices by 1 (and again defining X_0 as B). The scaling in the x-direction is done by a factor of x_1 and the scaling in the y-direction is done by a factor of y_1. We can therefore express the expected area of S as follows:

$$\mathbb{E}(\text{Area}(S)) = \mathbb{E}(\text{Area}(\triangle BY_1X_1)) + x_1y_1\mathbb{E}(\text{Area}(S))$$

$$\mathbb{E}(\text{Area}(S)) = \mathbb{E}\left(\frac{1}{2}(1-x_1)(1-y_1)\right) + \mathbb{E}(x_1y_1\text{Area}(S))$$

By independence of each variable, we can factor them through the expectation to get:

$$\mathbb{E}(\text{Area}(S)) = \frac{1}{2}\mathbb{E}(1-x_1)\mathbb{E}(1-y_1) + \mathbb{E}(x_1)\mathbb{E}(y_1)\mathbb{E}(\text{Area}(S))$$

$$\mathbb{E}(\text{Area}(S)) = \frac{1}{8} + \frac{1}{4}\mathbb{E}(\text{Area}(S))$$

$$\mathbb{E}(\text{Area}(S)) = \boxed{\frac{1}{6}}$$

6. Call a set of integers S *balanced* if it has the following two properties:

   1. S contains at least two integers.
   2. For any two integers a and b in S, if $a+b$ is even, then $\frac{a+b}{2}$ is also in the set.

   How many subsets of the set $\{1, 2, 3, \ldots, 10\}$ are balanced?

6. 66

Solution: We first characterize the balanced sets. Let S be any balanced set of integers. Let us sort the integers in S in ascending order and denote them as $S = \{s_1, s_2, \ldots, s_n\}$, where $s_1 < s_2 < \cdots < s_n$. Now, consider any two consecutive integers s_i and s_{i+1} in S. $s_i + s_{i+1}$ cannot be even because if it were even, $\frac{s_i+s_{i+1}}{2}$ would be another element in S in between s_i and s_{i+1}, contradicting consecutivity of s_i and s_{i+1}. Then, s_i and s_{i+1} must have opposite parities. Now, let us consider any three consecutive integers s_i, s_{i+1}, and s_{i+2} in S. We know that s_i and s_{i+1} must have opposite parities, and that s_{i+1}, and s_{i+2} must have opposite parities, so we conclude that s_i and s_{i+2} must have the same parity. Then, $s_i + s_{i+2}$ is even, so $\frac{s_i+s_{i+2}}{2}$ is in S. But the only element of S between s_i and s_{i+2} is s_{i+1}, so we must have that $\frac{s_i+s_{i+2}}{2} = s_{i+1}$. Then, s_i, s_{i+1}, s_{i+2} forms an arithmetic sequence. Since this is true for all triplets of consecutive integers in S, we have that $s_1, s_2, \ldots, s_n$ forms an arithmetic sequence. Finally, we note that since consecutive integers have opposite parity, the common difference must be odd. Therefore, we conclude that all balanced sets form arithmetic sequences with odd difference. Conversely, we can easily see that all sets that form arithmetic sequences with odd difference are balanced. Therefore, the problem reduces to finding the number of subsets of $\{1, 2, 3, \ldots, 10\}$ that form arithmetic sequences with odd difference. To count this, we do casework on the difference and the length of the arithmetic sequence. Consider the following table, where the entry at row r and column c denotes the number of arithmetic sequences with common difference r and length c.

	2	3	4	5	6	7	8	9	10
1	9	8	7	6	5	4	3	2	1
3	7	4	1						
5	5								
7	3								
9	1								

Summing up all the cases, we get $\boxed{66}$.

7. Let a, b, c, d, e, f be integers satisfying

$$a^2 + b^2 + c^2 = 9, \qquad d^2 + e^2 + f^2 = 225, \qquad ad + be + cf = 45.$$

   How many ordered sextuples (a, b, c, d, e, f) satisfy these conditions?

7. 30

Solution: By the Cauchy–Schwarz inequality,

$$(ad+be+cf)^2 \leq (a^2+b^2+c^2)(d^2+e^2+f^2).$$

Substituting the given values gives

$$(ad+be+cf)^2 = 45^2 = 2025$$

and

$$(a^2+b^2+c^2)(d^2+e^2+f^2) = 9 \cdot 225 = 2025.$$

Thus equality holds in Cauchy–Schwarz. Therefore the vectors (a,b,c) and (d,e,f) are proportional, so

$$(d,e,f) = t(a,b,c)$$

for some real number t.

Substituting into the third equation, we get

$$t(a^2+b^2+c^2) = 45$$

substituting in the first equation

$$t \cdot 9 = 45$$

so

$$t = 5.$$

It remains to count integer triples (a,b,c) satisfying

$$a^2+b^2+c^2 = 9.$$

The only two ways to express 5 as a sum of three squares of integers is

$$5 = 3^2+0^2+0^2 \qquad 5 = 2^2+2^2+1^2.$$

Thus (a,b,c) must be a permutation of $(\pm 3,0,0)$ or $(\pm 2,\pm 2,\pm 1)$.

There are 3 permutations of $(\pm 3,0,0)$ and 2 choices of sign of ± 3. There are 3 permutations of $(\pm 2,\pm 2,\pm 1)$ and 2^3 choices of signs. Therefore the number of triples (a,b,c) is

$$3 \cdot 2 + 3 \cdot 2^3 = 30.$$

For each such triple, (d,e,f) is uniquely determined as $3(a,b,c)$.

Hence the total number of ordered sextuples is

$$\boxed{30}.$$

8. Let c_n be the number of ways to place the integers from 1 through n^2, inclusive, in an n by n grid such that every row and every column forms an arithmetic sequence. What is $c_1 + c_2 + \ldots + c_{100}$?

8. 809

Solution: First, any configuration for $n = 1$ and $n = 2$ works, so we have $c_1 = 1$ and $c_2 = 24$. We claim that $c_n = 8$ for all $n \geq 3$.

Notice that by the required condition, the smallest and largest values, 1 and n^2, must be in a corner. Without the loss of generality, assume 1 is in the top left corner. We can multiply the number of valid cases by 4 in the end.

Now, consider where we can place 2. Notice that it has to be adjacent to 1, since there are no values between 1 and 2. Without the loss of generality, let it be right below 1. Then, the first column must be 1 through n. We can multiply the number of valid cases by 2 in the end.

Now, let's consider where we can place $n + 1$. It has to be in the second column, because all values smaller than $n + 1$ are already in the first column, so there would be no value to fill in the gap between $n + 1$ and the first column if it was not in the second column. In the second column, $n + 1$ must be in one end, because otherwise, at least one side would have a smaller element, which is a contradiction.

Case 1: $n + 1$ is at the bottom next to n

Then, the bottom row is $n + 1$ through $2n$. This would force n^2 to go on the top right corner, and force the top row to be $1, n + 2, 2n + 3, \ldots, n^2$. However, for $n \geq 3$, we know $n + 2$ is already in the bottom row, giving us a contradiction. Hence, 0 ways for this case.

Case 2: $n + 1$ is at the top next to 1

Then, the top row is $1, n+1, 2n+1, \ldots, (n-1)n+1$. This would force n^2 to go on the bottom right corner. This would give a valid placement of

$$\begin{bmatrix} 1 & n+1 & \ldots & (n-1)n+1 \\ 2 & n+2 & \ldots & (n-1)n+2 \\ \vdots & \vdots & \ldots & \vdots \\ n & 2n & \ldots & n^2 \end{bmatrix}$$

This case hence gives us $4 \times 2 \times 1 = 8$ valid configurations.

Hence, we have proven $c_n = 8$ for $n \geq 3$. Now, $c_1 + c_2 + \ldots c_{100} = 1 + 24 + 98 \times 8 = 809$.

9. Let S be a set of integers with the property that for any two distinct integers m and n in S, we have that $mn^{254} - nm^{254}$ is not divisible by 2026. What is the maximum possible number of integers that can be contained in S?

9. 4

Solution: First, we note that $mn^{254} - nm^{254} = mn(n^{253} - m^{253})$ is always even. This is because if m and n are both odd, then $n^{253} - m^{253}$ must be even.

Therefore, the condition reduces to $mn^{254} - nm^{254}$ not being divisible by 1013, so we have $mn(n^{253} - m^{253}) \not\equiv 0 \pmod{1013}$. This implies that m, n, $n^{253} - m^{253} \not\equiv 0 \pmod{1013}$. The first two inequivalencies only tell us that integers that are 0 mod 1013 cannot be in S.

Let us now focus on the last inequivalency: $n^{253} - m^{253} \not\equiv 0 \pmod{1013}$. This is equivalent to $n^{253} \not\equiv m^{253} \pmod{1013}$. By the pigeon-hole principle, this tells us that the maximum number of integers in S is equal to the total number of unique possible values of $n^{253} \pmod{1013}$ over all integers n.

We start by noticing that $4 \cdot 253 = 1012 = 1013 - 1$. By Fermat's Little Theorem, $n^{1012} \equiv 1 \pmod{1013}$. Note that 1 and -1 are the only possible values for $n^{506} \pmod{1013}$ because there can be at most 2 square roots of an integer under a prime modulus. If $n^{506} \equiv 1 \pmod{1013}$, then we can use a similar argument to see that 1 and -1 are the only possible values for $n^{253} \pmod{1013}$. If $n^{506} \equiv -1 \pmod{1013}$, we note that $45^2 = 2025 = 2026 - 1 = 1013 * 2 - 1$, so $45^2 \equiv -1 \pmod{1013}$. Then, using the same argument again, we see that 45 and -45 are the only possible values for $n^{253} \pmod{1013}$. We conclude that there are at most 4 possible values for n^{253}: 1, -1, 45, and -45. Each of these are indeed possible since $1^{253} \equiv 1$, $(-1)^{253} \equiv -1$, $45^{253} \equiv 45$, $(-45)^{253} \equiv -45 \pmod{1013}$. Therefore, the final answer is $\boxed{4}$.

Note: To see why there can be at most 2 square roots of an integer under a prime modulus p (and that they are in fact additive inverses of each other), let us suppose that $a^2 \equiv b^2$ (mod p). It follows that $(a-b)(a+b) \equiv 0$ (mod p). Since p is prime, by the zero-product-property, $a - b \equiv 0$ or $a + b \equiv 0$ (mod p), which implies that $a \equiv \pm b$ (mod p), as desired.

10. Given a parabola S, the *scissor parabola* of S, denoted $f(S)$, is defined as the locus of points of intersection of pairs of perpendicular lines that also each have a perpendicular point of intersection with S. Note that the scissor parabola of a parabola is always a parabola. Find the vertex of the parabola $f(f(f(S)))$ when S is the parabola defined by the equation $28y = x^2$.

10. $(0, \frac{441}{16})$

Solution: Intended Solution: First, consider the tangent to the general parabola $4ay = x^2$ with a slope m. Let the equation of such a tangent be $y = mx + b$.

The solution below is still for the original problem of $y^2 = 28x$, not $28y = x^2$. The problem has rotational symmetry though, so we would just have to rotate the answer below 90° counterclockwise.

Substituting into the equation of the parabola, we get

$$(mx + b)^2 = 4ax$$

Simplifying into a quadratic in x,

$$m^2x^2 + (2mb - 4a)x + b^2 = 0$$

Since there is a unique point of tangency, the discriminant must equal zero. Thus,

$$(2mb - 4a)^2 = 4m^2b^2 \implies b = \frac{a}{m}$$

Thus, the equation of a tangent with slope m is $y = mx + \frac{a}{m}$. Plugging $b = \frac{a}{m}$ back into the quadratic in x, we obtain the point of contact

$$x = \frac{a}{m^2} \implies y = \sqrt{4\frac{a^2}{m^2}} = \frac{2a}{m}$$

Now, the normal at this point will have a slope $m' = -\frac{1}{m}$. Hence, the point of contact in terms of m' is $(am'^2, -2am')$. We then obtain the equation of the normal with slope m':

$$y = m'x - 2am' - am'^3$$

or, as a cubic in m',

$$am'^3 + (2a - x)m' + y = 0$$

This implies that there are up to three normals that can be drawn to a parabola from a point. Let the corresponding slopes be m_1, m_2, and m_3, which are also roots of the cubic. Then, we have

$$m_1m_2m_3 = -\frac{y}{a}$$

$$m_1 + m_2 + m_3 = 0$$

$$m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a - x}{a}$$

Since we are looking for the locus of orthogonal normals, we can set $m_1m_2 = -1$. This gives us

$$m_3 = \frac{y}{a} = -(m_1 + m_2)$$

$$-1 - (m_1 + m_2)^2 = \frac{2a - x}{a}$$

Simplifying this pair of equations gives us the required locus, i.e. the equation of the scissor curve

$$y^2 = a(x - 3a) = 4\frac{a}{4} \cdot (x - 3a)$$

Thus, the scissor curve is a parabola with $\frac{1}{4}^{th}$ the focal length of the original parabola with the vertex shifted rightwards by three focal lengths. Iterating this n times gives us the equation of the n^{th} Scissor Curve S_n:

$$y^2 = 4 \cdot \frac{a}{4^n} \cdot (x - 3(a + \frac{a}{4} + \frac{a}{4^2} + \cdots + \frac{a}{4^{n-1}}))$$

Using our values $a = 7$ and $n = 3$, we get the equation of S_3:

$$y^2 = \frac{7}{16} \cdot (x - \frac{441}{16})$$

Thus, the vertex of S_3 is $(\frac{441}{16}, 0)$.