Power Round

Names: ____

Team Name: ____

Instructions

1. Do not begin until instructed to by the proctor.
2. You will have 90 minutes to solve the problems during this round.
3. Your submission will be graded and assigned point values out of the total points possible per problem. Your total score will be the sum of the points you receive for each problem.
4. Submissions will be graded on correctness as well as clarity of proof. A proof with significant progress towards a solution may receive more credit than a correct answer with no justification.
5. **You may use the result of a previous problem in the proof of a later problem, even if you do not submit a correct solution to the referenced problem.** However, you may not use the result of a later problem in the proof of an earlier problem.
6. Please submit each part of each problem on a separate page. Write your team name, problem number, and page number clearly at the top of each page.
7. No calculators or electronic devices are allowed.
8. All submitted work must be the work of your own team. You may collaborate with your team members, but no one else.
9. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
10. Do not discuss the problems with anyone outside of your team until all papers have been collected.
11. If you have a question or need to leave the room for any reason, please raise your hand quietly.
12. Good luck!

Acceptable Answers

1. Solutions should be written in proof format. All answers, reasoning, and deductions must be explained and justified, unless the problem explicitly asks for you to "compute". Problems asking you to "show", "prove", or "justify" **require proof!**
2. Proofs will be graded both on correctness as well as clarity of presentation.
3. Partial credit may be awarded for significant progress towards a solution.
4. Each problem must be written starting on a new, blank page. Two different problems should not be written on the same page.
5. At the top right corner of each page, please clearly print your team name, problem number, and page number.
6. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

Power Round

Problem 1

The Fibonacci sequence is a sequence denoted $f(0), f(1), f(2), \ldots$ with $f(0) = 0$, $f(1) = 1$, and $f(i) = f(i-2)+f(i-1)$ for all $i \geq 2$. For a positive integer n, the n-Fibonacci sequence, denoted $f_n(0), f_n(1), f_n(2), \ldots$ is defined in the same way, except for every term, we additionally take its remainder mod n. For example, $f(4) = 3$, so $f_2(4) = 1$, since the remainder when 3 is divided by 2 is equal to 1.

(a) (1 Point) List the values of $f_{10}(0), f_{10}(1), f_{10}(2), \ldots, f_{10}(10)$.

(b) (2 Point) Prove that for all positive integers n, there must exist distinct i and j such that $f_n(i) = f_n(j)$.

(c) (2 Points) Prove that for all positive integers n, there must exist distinct i and j such that $f_n(i) = f_n(j)$ and $f_n(i+1) = f_n(j+1)$. You may find it helpful *not* to use part (b) for this problem.

(d) (4 Points) Prove that for all positive integers n, if a term appears in the n-Fibonacci sequence, then it must appear another time in the n-Fibonacci sequence.

(e) (3 Points) Prove that for all positive integers n, if a term appears in the n-Fibonacci sequence, then it must appear infinitely many times in the n-Fibonacci sequence.

(f) (2 Points) Prove that for all positive integers n, there exists infinitely many terms of the regular Fibonacci sequence that are divisible by n.

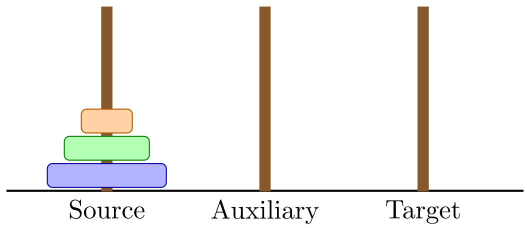
(g) (2 Points) Is the result in part (f) still true if we allow the first two terms of the Fibonacci sequence to be any pair of integers? Provide a brief justification if yes or a counterexample if not.

(h) (5 Points) The result in part (f) can be expanded to other Fibonacci-like sequences. Let $a_0, a_1, \ldots, a_{k-1}$ and $c_0, c_1, \ldots, c_{k-1}$ be integer constants. Define the sequence $g(0) = a_0, g(1) = a_1, \ldots, g(k-1) = a_{k-1}$, and $g(i) = c_0 g(i-k) + c_1 g(i-k+1) + \ldots + c_{k-1} g(i-1)$ for all $i \geq k$. If $g(0) = a_0 = 0$ and $c_0 = 1$, prove that for all positive integers n, there exists infinitely many terms of g that are divisible by n.

Problem 2.1

The Tower of Hanoi is a mathematical puzzle invented by Édouard Lucas in 1883. It consists of three pegs and several disks of distinct sizes, each of which can be placed on any peg. The goal is to move the entire stack from one peg to another while keeping the disks in increasing size order from top to bottom. The only rules are that exactly one disk may be moved at a time, and no larger disk may ever be placed on top of a smaller disk. Naturally, we ask: what is the minimum number of moves required?

Example with 3 disks:



(a) (1 Point) Let $H(n)$ be the minimum number of moves needed to move n disks from the first to the third peg, where initially it's placed in increasing size as we go down. Compute $H(1), H(2), H(3), H(4)$.

(b) (2 Points) Prove the recursive relationship $H(n) = 2H(n-1) + 1$ for $n \geq 2$.

(c) (2 Points) Find and prove the general formula for $H(n)$.

(d) (3 Points) Find and prove the general formula for if we only allow moving between adjacent pegs, i.e., between source and auxillary, and between auxillary and target.

Problem 2.2

What if instead of a "1-dimensional" stack, we consider a 2-dimensional grid? Consider the new problem.

Let there be three boxes, where second and third boxes are initially empty. The first box initially consists of 1 block of value 1, 2 blocks of value 2, $\dots$, $n-1$ blocks of value $n-1$, n blocks of value n, $n-1$ blocks of value $n+1$, $\dots$, 1 block of value $2n-1$. That is, v blocks of value $v \leq n$, and $2n - v$ blocks of value $v > n$. One way to visualize this is as an $n \times n$ grid of blocks whose values are the sums of the row and column indices minus 1, so the top-left value is 1 and the bottom-right value is $2n-1$.

1	2	3	$\cdots$	n
2	3	4	$\cdots$	$n+1$
3	4	5	$\cdots$	$n+2$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	$n+1$	$n+2$	$\cdots$	$2n-1$

Each turn, we are allowed to take one block of value v from one box into another if it's one of the blocks of least value in the original box, and all blocks in the receiving box have values $\geq v$, or if the receiving box is empty.

(a) (2 Point) Let $H_2(n)$ be the minimum number of turns needed to move all n^2 blocks from the first box to the third. Compute $H_2(1), H_2(2)$, and $H_2(3)$.

Let $T_n(v)$ be the minimum number of turns needed to move all blocks of value $\leq v$ from one box to another when we have n^2 blocks in the above configuration. Let $M_n(v)$ be the number of blocks of value v when we have n^2 blocks in the above configuration.

(b) (1 Point) Find a recursive formula of $T_n(v)$ in terms of $T_n(v-1)$ and $M_n(v)$. No proof needed.

(c) (2 Points) Prove the formula $H_2(n) = \sum_{v=1}^{2n-1} M_n(v)2^{2n-1-v}$.

(d) (4 Points) Using the fact that

$$M_n(v) = \begin{cases} v, & \text{if } 1 \leq v \leq n, \\ 2n - v, & \text{if } n < v \leq 2n-1, \end{cases}$$

prove that

$$H_2(n) = T_n(2n-1) = 4^n - 2^{n+1} + 1.$$

(e) (1 Point) Do you notice anything interesting between the formulas for $H(n)$ and $H_2(n)$?

Problem 2.3

Let's now generalize to an arbitrary d-dimensions. Let there be three d-dimensional boxes, where the second and third boxes are initially empty. The first box initially consists of a $n \times n \times \ldots n$ (d n's) grid of n^d blocks, each block having the value of the sum of their indices, where the indices start at 1 (so blocks start at value d now!). That is, the number of blocks of value v is the number of ways of $x_1 + x_2 + \ldots x_d = v$, where each x_i satisfies $1 \leq x_i \leq n$.

Let $H_d(n)$ be the minimum number of ways to move all such blocks from box 1 to box 3, subject to the constraint where we are allowed to take one block of value v from one box into another if it's one of the blocks of least value in the original box, and all blocks in the receiving box have values $\geq v$ (or if it's empty). Similarly, let $M_{d,n}(v)$ be the number of blocks of value v in the d-dimensional grid of sidelength n as defined above.

(a) (2 Points) Prove the formula $H_d(n) = \sum_{v=d}^{dn} M_{d,n}(v)2^{dn-v}$.

(b) (5 Points) Prove that $H_d(n) = H(n)^d = (2^n - 1)^d$.