Individual Round

Name: ______________________________

Team Name: ______________________________

Instructions

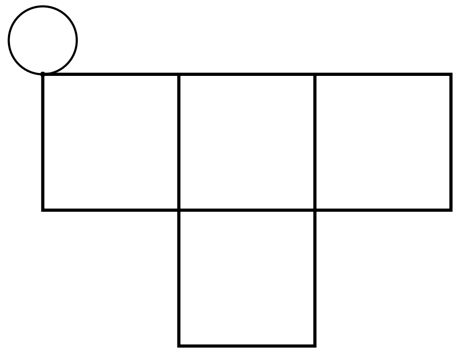
1. Do not begin until instructed to by the proctor.
2. You will have 60 minutes to solve 10 problems.
3. Your score will be the number of correct answers. There is no penalty for guessing or incorrect answers.
4. No calculators or electronic devices are allowed.
5. All submitted work must be your own. You may not collaborate with anyone else during the individual round.
6. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
7. Do not discuss the problems until all papers have been collected.
8. If you have a question or need to leave the room for any reason, please raise your hand quietly.
9. Good luck!

Acceptable Answers

1. All answers must be simplified as much as reasonably possible. For example, acceptable answers include $\sin(1^\circ)$, $\sqrt{43}$, or π^2. Unacceptable answers include $\sin(30^\circ)$, $\sqrt{64}$, or 3^2.
2. All answers must be exact. For example, π is acceptable, but 3.14 or 22/7 is not.
3. All rational, non-integer numbers must be expressed in reduced form $\pm\frac{p}{q}$, where p and q are relatively prime positive integers and $q \neq 0$. For example, $\frac{2}{3}$ is acceptable, but $\frac{4}{6}$ is not.
4. All radicals must be fully reduced. For example, $\sqrt{24}$ is not acceptable, and should be written as $2\sqrt{6}$. Additionally, rational expressions cannot contain radicals in the denominator. For example, $\frac{1}{\sqrt{2}}$ is not acceptable, and should be written as $\frac{\sqrt{2}}{2}$.
5. Answers should be expressed in base 10 unless otherwise specified.
6. Complex numbers should be expressed in the form $a + bi$, where both a and b are written in a form compliant with the rules above. In particular, no complex denominators are allowed. For example, $\frac{1+2i}{1-2i}$ should be written as $-\frac{3}{5} + \frac{4}{5}i$ or $\frac{-3+4i}{5}$.
7. If a problem asks for all solutions, you may give the answers in any order. However, no credit will be given if any solution is missing or any solution is given but not correct.
8. Angle measurements should be given in radians unless otherwise specified.
9. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

Individual Round

1. A circle of radius 1 is rolled without slipping clockwise around the following shape consisting of 4 squares of side length 4 glued together in a T-shape. The circle starts with its center directly above the top left vertex of the shape. How many rotations (including partial rotations) will the circle make before it returns to its original position?



1. $\frac{18}{\pi} + \frac{3}{2}$

Solution: Out of the 10 external edges total, the left most, right most, top most, and bottom most edges are rolled over completely (6 edges total). The other 4 edges, surrounding the concave vertices, are rolled over for a length of $4 - 1 = 3$.

For each convex vertex the circle rolls over, it rolls an additional $\frac{1}{4}$ rotations. There are 6 such vertices.

Hence, in total, the circle rolls $\frac{6\cdot4+4\cdot3}{2\pi} + 6 \cdot \frac{1}{4} = \frac{18}{\pi} + \frac{3}{2}$ rotations.

2. Define $f(x) = x^5 - x^4 - 10x^3 + 11x^2 + 9x + 2026$, and let $g(x)$ be the reflection of $f(x)$ over the y-axis. $f(x)$ and $g(x)$ intersect at $(0, 2026)$ as well as 4 other points. Find the area of the quadrilateral formed by these 4 points.

2. 32

Solution: We know $g(x) = f(-x) = -x^5 - x^4 + 10x^3 + 11x^2 - 9x + 2026$, and $f(x) = g(x)$ when $f(x) - g(x) = 2x^5 - 20x^3 + 18x = 0$. Dividing by 2 and factoring gives $x(x^2 - 1)(x^2 - 9) = x(x-1)(x+1)(x-3)(x+3)$. Since $x^5 - 10x^3 + 9x = 0$ for $x = 0, \pm1, \pm3$, we can just focus on the rest of the polynomial for our coordinates. So $f(0) = 2026, f(\pm1) = 2036$, and $f(\pm3) = 2044$. Then, our 4 points are $(\pm1, 2036), (\pm3, 2044)$, which form a trapezoid with area $\boxed{32}$.

3. Anakin Gridwalker wants to walk on the coordinate plane from $(0, 0)$ to $(4, 5)$. Each minute, he can move either 1 unit to the right or 1 unit upward. Darth Vader, Anakin's antimatter clone, starts at $(4, 5)$ and exactly mirrors Anakin's movements: Darth Vader moves down whenever Anakin moves up, and Darth Vader moves left whenever Anakin moves right. If Anakin and Darth Vader ever come into contact, either by meeting at a vertex at the end of a minute or by traversing the same edge during

the same minute, they annihilate each other, which would be very bad. How many ways can Anakin complete this walk without annihilating himself?

3. 90

Solution: Without the restriction that Anakin must avoid meeting his clone, there are

$$\binom{9}{5} = 126$$

such paths.

We now count and subtract the paths on which Anakin and Darth Vader meet. Since Anakin and Darth Vader start at opposite ends of a 9-step path, they cannot meet at the end of a minute: after t minutes, Anakin has taken t steps while Darth Vader has taken $9-t$ steps along the same route, and $t = \frac{9}{2}$ is impossible. Thus, the only way they can annihilate each other is by traversing the same edge in opposite directions during the same minute.

This can happen only on the fifth move, when the two walkers are equally far in time from the endpoints. The unique edge whose endpoints add to $(4,5)$ is the vertical edge from $(2,2)$ to $(2,3)$, so Anakin must traverse this edge on move 5. Therefore, after 4 moves Anakin must be at $(2,2)$, which can happen in

$$\binom{4}{2} = 6$$

ways, since he must make 2 right moves and 2 up moves. His fifth move is then forced to be upward.

From $(2,3)$ to $(4,5)$, Anakin must make 2 right moves and 2 up moves, which can be done in

$$\binom{4}{2} = 6$$

ways. Hence the number of bad paths is

$$6 \cdot 6 = 36.$$

Therefore, the number of valid paths is

$$126 - 36 = \boxed{90}.$$

4. Let $A_1A_2A_3A_4A_5A_6A_7A_8$ and $B_1B_2B_3B_4B_5B_6B_7B_8$ be two concentric regular octagons such that B_1 lies on the midpoint of line segment A_1B_2. What is the ratio of the area of the larger octagon to the area of the smaller octagon?

4. $5 - 2\sqrt{2}$

Solution: The midpoint condition forces $B_1B_2B_3B_4B_5B_6B_7B_8$ to be the smaller octagon. Suppose it has side length 1, and set x equal to the side length of $A_1A_2A_3A_4A_5A_6A_7A_8$. Then, the desired ratio of areas is equal to x^2.

Consider triangle $A_1B_2A_2$. B_1 is the midpoint of line segment A_1B_2, so line segment A_1B_2 has length 2. By rotational symmetry, B_2 is the midpoint of line segment A_2B_3, so line segment A_2B_2 has length 1. The interior angle of a regular octagon is $\frac{3\pi}{4}$, so angle $B_1B_2B_3$ has measure $\frac{3\pi}{4}$. Then, angle $A_1B_2A_2$ has measure $\frac{\pi}{4}$. By law of cosines, $x^2 = 2^2 + 1^2 - 2(2)(1)\cos(\frac{\pi}{4}) = \boxed{5 - 2\sqrt{2}}$.

5. There are 100 rotatable handles lined up on a wall, labeled 1 through 100. They all initially point straight upwards. There are also 100 people, labeled 1 through 100. One by one, the n'th person goes up to each handle with a label that is a multiple of n and turns it 90 degrees clockwise from its current position. After all 100 people have completed this process, how many handles will be pointing straight downwards?

5. 43

Solution: Consider handle n. The number of times it is turned is $\tau(n)$, the number of factors of n. What the problem wants is exactly the number of positive integers from 1 to 100 such that $\tau(n) \equiv 2 \mod 4$.

Let the prime factorization of n be $p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_k^{e_k}$. Then, we know $\tau(n) = (e_1+1)(e_2+1)\ldots(e_k+1)$. The fact that $\tau(n) \equiv 2 \mod 4$ means that exactly one of $(e_i + 1)$ is even with one factor of 2, and all others are odd. To formalize this better, $e_i \equiv 1 \mod 4$, and all other es are even.

This means that $n = p^j \cdot m^2$, where p is a prime that does not divide m, and $j \equiv 1 \mod 4$. Now, we can case work on m to find our answer.

$m = 1:$

Then p can be any prime ≤ 100. There are 25 of them. For 2, we can have 2^1 and 2^5, and all the other ones can only have p^1. Hence, we have **26** counts in this case.

$m = 2:$

Then p can be any prime that's not 2 that is ≤ 25. There are **8** such primes. Only $j = 1$ is allowed. Also, since $2^5 = 32 > 25$, we only need to consider $j = 1$ from now on.

$m = 3:$

We have $p \leq 11,\ p \neq 3$. Hence, $p = 2, 5, 7, 11$, so **4** cases.

$m = 4:$

We have $p \leq 6,\ p \neq 2$. Hence, $p = 3, 5$, so **2** cases.

$m = 5:$

We have $p \leq 4,\ p \neq 5$. Hence, $p = 2, 3$, so **2** cases.

$m = 6:$

We have $p \leq 2,\ p \neq 2, 3$, so **0** cases.

$m = 7:$

We have $p \leq 2,\ p \neq 7$, so **1** case.

$m \geq 8:$

We have $p \leq 1$, so **0** cases.

In total, we have $26 + 8 + 4 + 2 + 2 + 0 + 1 + 0 = \boxed{43}$ cases.

6. Anakin Pathwalker starts in the top left square of a 2×10 grid. From there, he may move up, down, left, or right to adjacent squares provided he stays within the grid. He may not revisit any squares, including his starting square. A *complete path* is a path that continues until there are no legal moves remaining. How many different complete paths can Anakin walk?

6. 767

Solution: Let $f(n)$ denote the number of complete paths in a $2 \times n$ grid starting from a corner. The base case is $f(2) = 2$, and for $n \geq 3$ we have the recurrence

$$f(n) = 2f(n-1) + 1.$$

This is because if initially, Anakin went down, we would have a $2 \times (n-1)$ grid starting from the corner. If Anakin intially went right, we essentially have a $2 \times (n-1)$ grid because the only way we can reach the bottom left square is if the way to traverse the $2 \times (n-1)$ sub-grid ended at its bottom left. The only exception not included is when we went right, down, then left immediately. This gives us the $+1$.

Solving this recurrence gives

$$f(n) = 3 \cdot 2^{n-2} - 1.$$

Therefore,

$$f(10) = 3 \cdot 2^8 - 1 = \boxed{767}.$$

7. Let A and B be two points chosen uniformly and independently at random from the interior of the unit cube centered at the origin with faces parallel to the coordinate planes (the set $\{(x, y, z) : |x| < \frac{1}{2}, |y| < \frac{1}{2}, |z| < \frac{1}{2}\}$). What is the probability that line segment AB intersects the first octant (the set $\{(x, y, z) : x > 0, y > 0, z > 0\}$)?

7. $\frac{5}{16}$

Solution: We first note that by symmetry, the probabilities of intersecting each of the 8 octants are equal. Then, by linearity of expectation, the expected total number of octants that AB intersects is equal to 8 times our desired probability. Let us compute this expected value. The segment AB begins in the octant containing A. Each time a coordinate of A and B have opposite signs, the segment must cross the corresponding coordinate plane and enter a new octant. For each coordinate, the probability that A and B have opposite signs is $\frac{1}{2}$. Hence, by linearity of expectation, the expected number of additional octants entered is $3 \cdot \frac{1}{2} = \frac{3}{2}$, so the total expected value is $1 + \frac{3}{2} = \frac{5}{2}$. Then, our desired probability is $(\frac{1}{8})(\frac{5}{2}) = \boxed{\frac{5}{16}}$.

8. Velvet has 9 dogs with ages $1, 2, 3, ..., 9$. She divides her dogs into 3 nonempty groups and computes the total age of each group. How many ways can Velvet do this grouping such that the three total ages can be arranged to form an arithmetic sequence?

8. 453

Solution: Let the three group sums be a, b, c. Since they can be arranged to form an arithmetic sequence, we have

$$a + b + c = 1 + 2 + \cdots + 9 = 45,$$

so their average is 15. Thus the three sums must be

$$15 - d, \quad 15, \quad 15 + d$$

for some integer $d \geq 0$. In particular, at least one of the groups must have total age 15. We split into two cases.

Case 1: The three sums are not all equal.

Then there is a unique group whose sum is 15. Choose this group and call it M. Once M is fixed, the remaining dogs must be split into two groups with sums less than and greater than 15, so it is enough to count subsets of the complement of M whose sum is less than 15.

The subsets of $\{1, 2, \ldots, 9\}$ with sum 15 are

$$\{6,9\}, \{7,8\}, \{1,5,9\}, \{1,6,8\}, \{2,4,9\}, \{2,5,8\}, \{2,6,7\}, \{3,4,8\}, \{3,5,7\}, \{4,5,6\},$$

$$\{1,2,3,9\}, \{1,2,4,8\}, \{1,2,5,7\}, \{1,3,4,7\}, \{1,3,5,6\}, \{2,3,4,6\}, \{1,2,3,4,5\}.$$

For each possible choice of M, the number of valid choices for the smaller-sum group is as follows:

M	number of choices
$\{6,9\}, \{7,8\}$	59 each
$\{1,5,9\}, \{2,4,9\}, \{3,4,8\}, \{3,5,7\}$	29 each
$\{1,6,8\}, \{2,5,8\}, \{2,6,7\}, \{4,5,6\}$	30 each
$\{1,2,3,9\}, \{1,2,4,8\}, \{1,2,5,7\}, \{1,3,4,7\}, \{1,3,5,6\}, \{2,3,4,6\}$	14 each
$\{1,2,3,4,5\}$	6

So the number of groupings in this case is

$$2 \cdot 59 + 4 \cdot 29 + 4 \cdot 30 + 6 \cdot 14 + 6 = 444.$$

Case 2: All three sums are equal.

Then each group must have sum 15. The 9 ways to partition $\{1, 2, \ldots, 9\}$ into three groups each summing to 15 are:

$$\{1,2,3,4,5\}, \{6,9\}, \{7,8\},$$

$$\{1,2,3,9\}, \{4,5,6\}, \{7,8\},$$

$$\{1,2,4,8\}, \{3,5,7\}, \{6,9\},$$

$$\{1,2,5,7\}, \{3,4,8\}, \{6,9\},$$

$$\{1,3,4,7\}, \{2,5,8\}, \{6,9\},$$

$$\{1,3,5,6\}, \{2,4,9\}, \{7,8\},$$

$$\{1,5,9\}, \{2,3,4,6\}, \{7,8\},$$

$$\{1,5,9\}, \{2,6,7\}, \{3,4,8\},$$

$$\{1,6,8\}, \{2,4,9\}, \{3,5,7\}.$$

Hence this case contributes 9 more groupings.

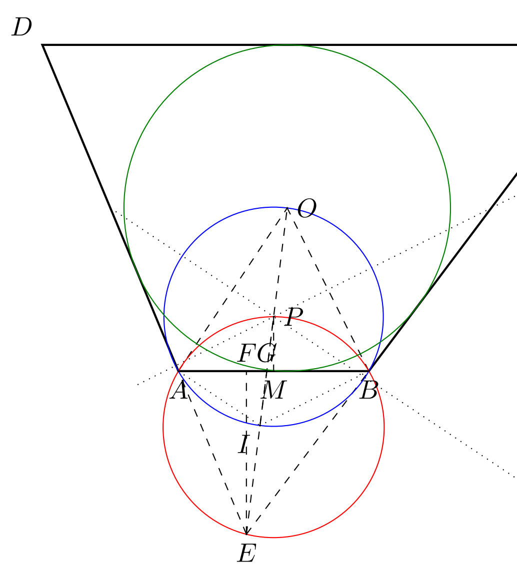
Therefore, the total number of valid groupings is

$$444 + 9 = \boxed{453}.$$

9. Let $ABCD$ be a trapezoid with $\overline{AB} \parallel \overline{CD}$ where $AB = 14, CD = 42$, and $AD = 26$. A circle with center O is internally tangent to every side of $ABCD$. If the perpendicular bisectors of $\overline{AO}$ and $\overline{BO}$ intersect at a point P, what is the distance from P to $\overline{AB}$?

9. **4**

Solution:



First, since $ABCD$ has a circle that's internally tangent to all sides, we know that $AB + CD = BC + AD$ (since we can drop all tangents and match the lengths). Plugging in what we are given, we get that $BC = 30$.

Extend $\overline{BC}$ and $\overline{AD}$ to meet at a point E. Notice that $\overline{AO}$ bisects $\angle BAD$ and $\overline{BO}$ bisects $\angle ABC$. This means that O is the E-excenter of $\triangle EAB$. Let I be the incenter of $\triangle EAB$. Then, by the incenter-excenter lemma, I, A, O, B are concyclic with diameter $\overline{IO}$ and center the midpoint of arc $\widehat{AB}$ of the circumcircle of $\triangle EAB$.

Now, let ℓ_1 and ℓ_2 be the perpendicular bisectors of segments $\overline{AO}$ and $\overline{BO}$ respectively. Since $\overline{AO}$ and $\overline{BO}$ are both chords of the circle containing I, A, O, B, they both pass through its center, and hence intersect there. Hence, P is the center of the circle of I, A, O, B, which is the midpoint of arc $\widehat{AB}$ in the circumcircle of $\triangle EAB$.

Notice that since $\overline{AB} \parallel \overline{CD}$, so $\triangle EAB \sim \triangle ECD$. They have a ratio of 3, so $EA = 13, EB = 15$. This means that $\triangle EAB$ is a $13-14-15$ triangle, so it has area 84 by dropping the height from E and splitting it into $5-12-13$ and $9-12-15$ right triangles.

Let M be the midpoint of segment $\overline{AB}$. What we want is then the length PM, since $\overline{PM} \perp \overline{AB}$. Let the foot of the height from E onto $\overline{AB}$ be F, and let $\overline{EP}$ intersect $\overline{AB}$ at G. Notice that $\triangle PMG \sim \triangle EFG$ with ratio $\frac{MG}{FG}$.

By the angle-bisector theorem, $AG = AB \cdot \frac{EA}{EA+EB} = 14 \cdot \frac{13}{13+15} = \frac{13}{2}$. This means that $MG = AM - AG = 7 - \frac{13}{2} = \frac{1}{2}$, and $FG = AG - AF = \frac{13}{2} - 5 = \frac{3}{2}$, since $\triangle AFE$ is a $5-12-13$ triangle.

This tells us that $\frac{MG}{FG} = \frac{1}{3}$, so $PM = \frac{1}{3} \cdot EF = \frac{1}{3} \cdot 12 = \boxed{4}$.

10. Kermit the Frog wants to visit every lily pad on Lilyline Pond, which has 7 lily pads lined up in a row from left to right. Kermit starts by choosing one of the 7 lily pads uniformly at random to begin on. Then, Kermit makes 6 jumps one by one, each time jumping uniformly at random to a lily pad that he has not yet visited. What is the probability that Kermit makes exactly two jumps to the left?

10. $\frac{397}{1680}$

Solution: Let us solve this in the general case with n lily pads. Label the lily pads $1, 2, \ldots, n$ from left to right. Let us first compute the number of jumping sequences with exactly 1 left jump. Consider the process of partitioning $\{1, 2, \ldots, n\}$ into two subsets. There are 2^n ways to do this. We turn this into a jumping sequence by having Kermit jump onto the first subset's lily pads in ascending order, then the second subset's lily pads in ascending order. For example, when $n = 6$, the partition $\{2, 4, 7\}, \{1, 3, 5, 6\}$ would correspond to the jumping sequence $2, 4, 7, 1, 3, 5, 6$. We note that this always produces exactly 1 left jump except in the case where the largest element in the first subset is less than the smallest element in the second subset (or when either subset is empty), which occurs $n+1$ times. Then, the number of jumping sequences with exactly 1 left jump is equal to $2^n - (n+1)$. Denote this as m.

Now, we compute the number of jumping sequences with exactly 2 left jumps. Consider the process of partitioning $\{1, 2, \ldots, n\}$ into three subsets. There are 3^n ways to do this. We turn this into a jumping sequence by having Kermit jump onto the first subset's lily pads in ascending order, then the second subset's lily pads in ascending order, then finally the third subset's lily pads in ascending order. For example, when $n = 7$, the partition $\{2, 4, 7\}, \{1, 3, 6\}, \{5\}$ would correspond to the jumping sequence $2, 4, 7, 1, 3, 6, 5$. We note that this always produces exactly 2 left jumps except when any of these (possibly overlapping) cases are true:

Case 1: The largest element in the first subset is less than the smallest element in the second subset.

Case 2: The largest element in the second subset is less than the smallest element in the third subset.

Case 3: Any of the subsets are empty.

Using the convention that the maximum element of the empty set is -infinity and the minimum element of the empty set is infinity, we can incorporate case 3 into cases 1 and 2.

Using stars and bars, there are $\binom{n+2}{2}$ partitions where both case 1 and case 2 are true.

It remains to compute the number of partitions where exactly one of case 1 or case 2 is true. Note that these partitions correspond to jumping sequences with exactly 1 left jump. However, we overcount by a factor of $n+1$ because multiple partitions can produce the same jumping sequence (e.g. $\{1,4,7\},\{2,3\},\{5,6\}$ and $\{1\},\{4,7\},\{2,3,5,6\}$ and $\{\},\{1,4,7\},\{2,3,5,6\}$ all produce the same jumping sequence of 1, 4, 7, 2, 3, 5, 6). In other words, each jumping sequence with exactly one left jump arises from exactly $n+1$ such partitions, corresponding to the $n+1$ possible placements of the third boundary, so these partitions are counted $(n+1)$-to-1. Therefore, the number of partitions where exactly one of case 1 or case 2 is true is equal to $(n+1)m$.

Therefore, the number of jumping sequences with exactly 2 left jumps is equal to $3^n - \left[\binom{n+2}{2} + (n+1)m\right]$.

Plugging in $n = 7$, we get $m = 2^7 - (7+1) = 120$, and $3^7 - \left[\binom{7+2}{2} + (7+1)(120)\right] = 1191$. There are $7! = 5040$ total jumping sequences, so the desired probability is $\frac{1191}{5040} = \boxed{\frac{397}{1680}}$.