Tiebreaker Round

Name: _____

Team Name: _____

Instructions

1. You will have 20 minutes to solve 3 problems. Do not begin until instructed to by the proctor.
2. When you would like to submit your answers, please inform your proctor.
3. **Your score will be the number of correct answers, with ties broken by time of submission.**
4. No calculators or electronic devices are allowed.
5. All submitted work must be your own. You may not collaborate with anyone else during the individual round.
6. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
7. Do not discuss the problems until all papers have been collected.
8. If you have a question or need to leave the room for any reason, please raise your hand quietly.
9. Good luck!

Acceptable Answers

1. All answers must be simplified as much as reasonably possible. For example, acceptable answers include $\sin(1^\circ)$, $\sqrt{43}$, or π^2. Unacceptable answers include $\sin(30^\circ)$, $\sqrt{64}$, or 3^2.
2. All answers must be exact. For example, π is acceptable, but 3.14 or 22/7 is not.
3. All rational, non-integer numbers must be expressed in reduced form $\pm\frac{p}{q}$, where p and q are relatively prime positive integers and $q \neq 0$. For example, $\frac{2}{3}$ is acceptable, but $\frac{4}{6}$ is not.
4. All radicals must be fully reduced. For example, $\sqrt{24}$ is not acceptable, and should be written as $2\sqrt{6}$. Additionally, rational expressions cannot contain radicals in the denominator. For example, $\frac{1}{\sqrt{2}}$ is not acceptable, and should be written as $\frac{\sqrt{2}}{2}$.
5. Answers should be expressed in base 10 unless otherwise specified.
6. Complex numbers should be expressed in the form $a + bi$, where both a and b are written in a form compliant with the rules above. In particular, no complex denominators are allowed. For example, $\frac{1+2i}{1-2i}$ should be written as $-\frac{3}{5} + \frac{4}{5}i$ or $\frac{-3+4i}{5}$.
7. If a problem asks for all solutions, you may give the answers in any order. However, no credit will be given if any solution is missing or any solution is given but not correct.
8. Angle measurements should be given in radians unless otherwise specified.
9. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

Tiebreaker Round

1. In base ten, 6245^2 is an eight-digit number. What are its middle four digits?

1. 0000

Solution: Since 6245 is a multiple of 5, its square ends in 25, which we don't need for the answer. By difference of squares, $6245^2 - 5^2 = 6240 \cdot 6250$. We can factorize this as $(10 \cdot 2^4 \cdot 39) \cdot (10 \cdot 5^4) = 10^6 \cdot 39$, so the middle four digits are $\boxed{0000}$.

2. In a bag, there are 7 red marbles, 2 green marbles, and 1 blue marble. You repeatedly take one marble at a time out of the bag uniformly at random without replacement until you take out the blue marble. What is the probability that one of the marbles you took out was green?

2. $\frac{2}{3}$

Solution: This is equivalent to the probability that in a random permutation of the 10 marbles, a green marble occurs before the blue marble. Note that by symmetry, we can ignore the red marbles. Then, we just need to compute the probability that in a random permutation of 2 green marbles and 1 blue marble, a green marble occurs before the blue marble. This is easily seen to be $\boxed{\frac{2}{3}}$.

3. Construct a circle of radius r, and let A and B be points on the circumference of the circle such that the line segment AB forms a diameter of the circle. Construct a chord of the circle that has length 8 and is perpendicular to line segment AB. Suppose there is a point C lying on the chord such that $\angle ABC = 45^\circ$ and the length of line segment AC is equal to 7. What is r?

3. $\frac{9}{2}$

Solution: Let P be the intersection point of AB and the chord. Let x equal the length of AP, and let y equal the length of BP. AB and the chord are perpendicular, so $\angle APC = \angle BPC = 90^\circ$. Since $\angle ABC = 45^\circ$, triangle BPC is a $45^\circ - 45^\circ - 90^\circ$ triangle, so the length of PC is equal to the length of BP, which is equal to y. Triangle APC is a right triangle, so we have $x^2 + y^2 = 7^2 = 49$. Also, a diameter splits a chord exactly in half, so by power of a point, $xy = 4^2 = 16$. Then, $(x+y)^2 = x^2 + y^2 + 2xy = 49 + 2 \cdot 16 = 81$, so $x + y = \sqrt{81} = 9$. $x + y$ is just the length of AB, a diameter of the circle, so $r = \frac{x+y}{2} = \boxed{\frac{9}{2}}$.