Team Round

Names: ______________________________

Team Name: ______________________________

Instructions

1. Do not begin until instructed to by the proctor.
2. You will have 60 minutes to solve 10 problems. Your score will be the number of correct answers. There is no penalty for guessing or incorrect answers.
3. **Only the official team answers will be graded.** If you are submitting the official answer sheet for your team, indicate this by writing "(OFFICIAL)" next to your team name. Do not submit any unofficial answer sheets.
4. No calculators or electronic devices are allowed.
5. All submitted work must be the work of your own team. You may collaborate with your team members, but no one else.
6. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
7. Do not discuss problems with anyone outside of your team until all papers have been collected.
8. If you have a question or need to leave the room for any reason, please raise your hand quietly.

Acceptable Answers

1. All answers must be simplified as much as reasonably possible. For example, acceptable answers include $\sin(1^\circ)$, $\sqrt{43}$, or π^2. Unacceptable answers include $\sin(30^\circ)$, $\sqrt{64}$, or 3^2.
2. All answers must be exact. For example, π is acceptable, but 3.14 or 22/7 is not.
3. All rational, non-integer numbers must be expressed in reduced form $\pm\frac{p}{q}$, where p and q are relatively prime positive integers and $q \neq 0$. For example, $\frac{2}{3}$ is acceptable, but $\frac{4}{6}$ is not.
4. All radicals must be fully reduced. For example, $\sqrt{24}$ is not acceptable, and should be written as $2\sqrt{6}$. Additionally, rational expressions cannot contain radicals in the denominator. For example, $\frac{1}{\sqrt{2}}$ is not acceptable, and should be written as $\frac{\sqrt{2}}{2}$.
5. Answers should be expressed in base 10 unless otherwise specified.
6. Complex numbers should be expressed in the form $a + bi$, where both a and b are written in a form compliant with the rules above. In particular, no complex denominators are allowed. For example, $\frac{1+2i}{1-2i}$ should be written as $-\frac{3}{5} + \frac{4}{5}i$ or $\frac{-3+4i}{5}$.
7. If a problem asks for all solutions, you may give the answers in any order. However, no credit will be given if any solution is missing or any solution is given but not correct.
8. Angle measurements should be given in radians unless otherwise specified.
9. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

Team Round

1. Consider a right triangle with the property that its side lengths form a geometric progression. What is the ratio of the triangle's longest side length to its shortest side length?

1. $\frac{1+\sqrt{5}}{2}$

Solution: Let x, cx, and c^2x be the side lengths of the triangle with $c \geq 1$ and $x > 0$. The longest side has to be the hypotenuse, so $x^2 + (cx)^2 = (c^2x)^2$. We can cancel the x^2 since $x > 0$, so $1 + c^2 = c^4$. Solving using the quadratic formula, we get $\frac{c^2x}{x} = c^2 = \boxed{\frac{1+\sqrt{5}}{2}}$.

2. In the expression (2?0?2?5)(2?0?2?5)(2?0?2?5)(2?0?2?5), each question mark is uniformly at random and independently replaced by either $+$ (a plus sign) or $-$ (a minus sign). What is the probability that the resulting expression equals 2025?

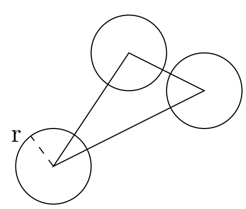
2. $\frac{3}{64}$

Solution: First of all, the ± 0's don't matter, so we just need to examine each term (2?2?5). Trying all combinations of $\pm$, this has a $1/4$ probability of taking on each of the values from $\{-5, -1, 5, 9\}$. Since $2025 = 9^2 \cdot 5^2$ and no other possibility can contribute factors of 3, we must have exactly two 9s. We then need two 5s or two -5s. This gives an answer of

$$2 \cdot \frac{1}{4^4}\binom{4}{2} = \boxed{\frac{3}{64}}$$

.

3. Let P be a convex 2025-sided polygon. Construct a total of 2025 circles of equal radius centered at each of the vertices of P such that none of the circles overlap. Let a equal the area within P that is covered by circles, and let A equal the total area of all 2025 circles. What is the value of $\frac{a}{A}$?



Example for a 3-sided polygon

3. $\frac{2023}{4050}$

Solution: Working in degrees, the sum of all angles of a n-sided polygon is given by $180(n-2)$. For a 2025-sided polygon, this is equal to $180 \cdot 2023$

The sum of all angles of the n circles is equal to $n \cdot 360$. For a 2025-sided polygon, there are 2025 vertices, so this is equal to $360 \cdot 2025$

Compute the fraction
$$\frac{180 \cdot 2023}{360 \cdot 2025} = \frac{2023}{2 \cdot 2025} = \frac{2023}{4050} \tag{1}$$

$\boxed{\frac{2023}{4050}}$ is a fraction of integers in its most reduced form, and is therefore the answer.

4. Find the sum of all positive integers k with the property that for any positive integer n with at least four digits, removing the fourth to last digit of n (i.e. $476389 \to 47389$) does not affect its remainder when divided by k.

4. 2340

Solution: We have for all a, b, c, d, e, $10000a + 1000b + 100c + 10d + e \equiv 1000a + 100c + 10d + e \pmod{k}$, so $9000a + 1000b \equiv 0 \pmod{k}$. This can only be true for all a and b if k is a factor of 1000. Then, the answer is the sum of all factors of 1000, which is equal to $(1+2+4+8)(1+5+25+125) = \boxed{2340}$.

5. Define the sets $S_1 = \{1, 2, 3, 4, 5, 6, 7\}$ and $S_2 = \{2, 4, 6, 8, 10, 12, 14\}$. How many surjective functions f from S_1 to S_2 are there such that for all n in S_1, the following is true: $\frac{1}{2}f(\frac{1}{2}f(\frac{1}{2}f(n))) = n$? Recall that "surjective" means that every element of S_2 has an element of S_1 that maps to it.

5. 351

Solution: Note it is equivalent to count surjective functions f from S_1 to S_1 such that $f(f(f(n))) = n$. For f to have this property, it must consist of cycles of 1 or 3 (the factors of 3). We will do casework on the number of 3-cycles.

Zero 3-cycles: There is 1 function – the identity function.

One 3-cycle: There are $\binom{7}{3}$ ways to choose 3 elements to be in the 3 cycle and 2 ways to orient the cycle (clockwise and counterclockwise). The remaining elements are sent to themselves. This totals to 70 functions.

Two 3-cycles: There are $\frac{1}{2}\binom{7}{3}\binom{4}{3}$ ways to choose two 3 element groups and $2 \cdot 2$ ways to orient them. The last element is sent to itself. This totals to 280 functions.

We have $1 + 70 + 280 = \boxed{351}$ functions total.

6. A semi-magic square is a square of numbers such that the sum of the numbers in each row and the sum of the numbers in each column are all the same. Note that this does not include the sum of the two diagonals. For example, the following is a semi-magic square because the sum of every column and the sum of every row is equal to 6:

0	3	3
3	2	1
3	1	2

How many 3×3 semi-magic squares exist where the nine numbers in the square are taken from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 11\}$ with each number appearing at most once.

6. 216

Solution: For every valid semi-magic square we can switch the order of the rows and columns and maintain that the square is semi-magic. This means that for each valid square we have there are $3! \cdot 3! - 1 = 35$ other valid squares that we will consider "equivalent".

We will first consider the square with $1, \ldots, 9$.

Note that the sum of each row/column must be 15 as $1 + \cdots + 9 = 45$. We will start constructing a valid magic square by placing a 9. If we place the 9, the locations of the 8 and 7 are "determined" (they are determined considering the equivalence of squares) as any pair of these cannot be in the same row/column. This leaves us with two locations for the 6 and after that the square is determined. In this example, the six can either go in either place marked by $-$

9		
	8	-
	-	7

This leaves us with $2 \cdot 36 = 72$ possible squares. We can also think of the two places for the six as the possibility of flipping rows and columns to get another equivalent square.

Note that if we want to replace one of the 1..9 with a 11 then what we replace has to be equivalent to 11 mod 3 as we need to divide the total sum by three for the sum for each of the three rows/each of the three columns. The possibilities for what to replace are now $2, 5, 8$. There are valid squares for replacing 2 and 8 but not 5.

11	4	3
1	9	8
6	5	7

11	1	4
2	9	5
3	6	7

This leaves us with $3 \cdot 72 = \boxed{216}$ squares.

7. Let $f(x)$ be the product of all unique rational numbers that can be written as $\frac{a}{a-2}$ such that a is an integer between 3 and x, inclusive.

Find the smallest positive integer z such that

$$\frac{\sqrt{2^{2025} \cdot \prod_{i=5}^{2025} f(i)}}{z} = n! \text{ where } n \text{ is an integer.}$$

7. 15

Solution: Let us first test some values x for $f(x)$ so we get an intuition of what the function does. $f(5) = \frac{5}{3} \cdot \frac{4}{2} \cdot \frac{3}{1} = \frac{5 \cdot 4}{2 \cdot 1}$, $f(6) = \frac{6}{4} \cdot \frac{5}{3} \cdot \frac{4}{2} \cdot \frac{3}{1} = \frac{6 \cdot 5}{2 \cdot 1}$, $f(7) = \frac{7}{5} \cdot \frac{6}{4} \cdot \frac{5}{3} \cdot \frac{4}{2} \cdot \frac{3}{1} = \frac{7 \cdot 6}{2 \cdot 1}$

Therefore we can generalize $f(x)$ to $f(x) = \frac{(x)(x-1)}{2} = \frac{(x-1)(x)}{2}$

Next, let us calculate $\prod_{i=5}^{2025} f(i) = f(5) \cdot f(6) \cdot f(7) \cdot ... \cdot f(2023) \cdot f(2024) \cdot f(2025)$. Using our generalized formula for $f(x) = \frac{(x-1)(x)}{2}$

$$\prod_{i=5}^{2025} f(i) = \frac{(4)(5)}{2} \cdot \frac{(5)(6)}{2} \cdot \frac{(6)(7)}{2} \cdot ... \cdot \frac{(2022)(2023)}{2} \cdot \frac{(2023)(2024)}{2} \cdot \frac{(2024)(2025)}{2} \tag{2}$$

Let us first take care of the 2 in the denominator. There are a total of $2025 - (5 - 1) = 2021$ terms in the product multiplied together. Therefore, we can write the 2 in the bottom as 2^{2021}.

Now let us take care of the numerator. Notice that every number 5 through 2024 shows up in the product twice. Therefore they can all be written as squared. 4 and 2025 only show up once. Therefore, the numerator can be written as $4 \cdot 5^2 \cdot 6^2 \cdot ... \cdot 2023^2 \cdot 2024^2 \cdot 2025$

Therefore, the product is equal to $\frac{4 \cdot 5^2 \cdot ... \cdot 2024^2 \cdot 2025}{2^{2021}}$ Plugging back into the left side of the equation, we get

$$\frac{\sqrt{2^{2025} \cdot \frac{4 \cdot 5^2 \cdot ... \cdot 2024^2 \cdot 2025}{2^{2021}}}}{z} = n! \tag{3}$$

Now simplify the left hand side by using exponent rules for $\frac{2^{2025}}{2^{2021}} = 2^4$ and moving all squared terms outside of the radical.

$$\frac{(5 \cdot 6 \cdot ... \cdot 2023 \cdot 2024)\sqrt{2^4 \cdot 4 \cdot 2025}}{z} = n! \tag{4}$$

Simplifying under the radical, note that $\sqrt{2^4} = 2^2 = 4$, $\sqrt{4} = 2$, and note that 2025 is a perfect square! $45^2 = 2025$. We are in a perfect square year how cooooooooool. We can now rewrite our equation as

$$\frac{(2 \cdot 4 \cdot 5 \cdot 6 \cdot ... \cdot 2023 \cdot 2024)(45)}{z} = n! \tag{5}$$

Note that $45 = 3 \cdot 15$

$$\frac{(2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot ... \cdot 2023 \cdot 2024)(15)}{z} = n! \tag{6}$$

Note that $(2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot ... \cdot 2023 \cdot 2024) = 2024!$

$$\frac{(2024!)(15)}{z} = n! \tag{7}$$

To minimize z, n must be 2024

$$\frac{(2024!)(15)}{z} = 2024! \qquad (8)$$

Thus $z = \boxed{15}$

8. The equation of an ellipse is given by $\frac{x^2}{7^2} + \frac{y^2}{11^2} = 1$. Let P be the locus of the point of intersection of a pair of perpendicular lines both tangent to the ellipse. Find the area enclosed within P.

8. 170π

Solution: Let the equation of any tangent to the ellipse be:

$$y = mx + c$$

Plugging into the general equation of the ellipse,

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$

Upon simplifying, we get a quadratic in x:

$$x^2(a^2m^2 + b^2) + 2a^2mcx + a^2(c^2 - b^2) = 0$$

Since $y = mx + c$ is tangent to the ellipse, the discriminant is zero.

$$(2a^2mc)^2 - 4a^2(c^2 - b^2)(a^2m^2 + b^2) = 0$$
$$c^2 - a^2m^2 - b^2 = 0$$
$$c = \pm\sqrt{a^2m^2 + b^2}$$

Therefore, the equation of a tangent with slope m is given by:

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

Squaring on both sides and rearranging,

$$m^2(x^2 - a^2) + m(-2xy) + (y^2 - b^2) = 0$$

For the two tangents represented by this equation to be perpendicular, we require:

$$\frac{y^2 - b^2}{x^2 - a^2} = -1$$
$$x^2 + y^2 = a^2 + b^2$$

Clearly, the locus is a circle of radius $\sqrt{a^2 + b^2}$. Thus, the area is given by $\pi(a^2 + b^2)$. In this problem, a = 7 and b = 11. Thus, the area enclosed is $\boxed{170\pi}$.

9. Solve for θ in the interval $[0, 2\pi]$:

$$\prod_{i=1}^{2025} \frac{2\sin\left(\frac{\theta}{2^{i-1}}\right)\cos\left(\frac{\theta}{2^i}\right)}{\sin\left(\frac{\theta}{2^{i-2}}\right)} = 1$$

You may leave exponents in your answer.

9. $\frac{2^{2026}\pi}{1+2^{2025}}$

Solution:

$$\prod_{i=1}^{2025} \frac{2\sin\left(\frac{\theta}{2^{i-1}}\right)\cos\left(\frac{\theta}{2^{i}}\right)}{\sin\left(\frac{\theta}{2^{i-2}}\right)} = 1$$

$$\implies \left(\prod_{i=1}^{2025} \frac{\sin\left(\frac{\theta}{2^{i-1}}\right)}{\sin\left(\frac{\theta}{2^{i-2}}\right)}\right)\left(\prod_{i=1}^{2025} 2\cos\left(\frac{\theta}{2^{i}}\right)\right) = 1$$

$$\implies \left(\frac{\sin\left(\frac{\theta}{2^{2024}}\right)}{\sin\left(\frac{\theta}{2^{-1}}\right)}\right)\left(\prod_{i=1}^{2025} 2\cos\left(\frac{\theta}{2^{i}}\right)\right) = 1 \qquad \text{by telescoping}$$

$$\implies \left(\frac{\sin\left(\frac{\theta}{2^{2025}}\right)\cos\left(\frac{\theta}{2^{2025}}\right)}{\sin(\theta)\cos(\theta)}\right)\left(\prod_{i=1}^{2025} 2\cos\left(\frac{\theta}{2^{i}}\right)\right) = 1 \qquad \text{by double angle formula}$$

$$\implies \left(\frac{2\cos\left(\frac{\theta}{2^{2025}}\right)\sin\left(\frac{\theta}{2^{2025}}\right)\cos\left(\frac{\theta}{2^{2025}}\right)}{\sin(\theta)\cos(\theta)}\right)\left(\prod_{i=1}^{2024} 2\cos\left(\frac{\theta}{2^{i}}\right)\right) = 1 \qquad \text{pulling out one term}$$

$$\implies \left(\frac{\sin\left(\frac{\theta}{2^{2024}}\right)\cos\left(\frac{\theta}{2^{2025}}\right)}{\sin(\theta)\cos(\theta)}\right)\left(\prod_{i=1}^{2024} 2\cos\left(\frac{\theta}{2^{i}}\right)\right) = 1 \qquad \text{by double angle formula}$$

$$\implies \left(\frac{2\cos\left(\frac{\theta}{2^{2024}}\right)\sin\left(\frac{\theta}{2^{2024}}\right)\cos\left(\frac{\theta}{2^{2025}}\right)}{\sin(\theta)\cos(\theta)}\right)\left(\prod_{i=1}^{2023} 2\cos\left(\frac{\theta}{2^{i}}\right)\right) = 1 \qquad \text{pulling out another term}$$

$$\implies \left(\frac{\sin\left(\frac{\theta}{2^{2023}}\right)\cos\left(\frac{\theta}{2^{2025}}\right)}{\sin(\theta)\cos(\theta)}\right)\left(\prod_{i=1}^{2023} 2\cos\left(\frac{\theta}{2^{i}}\right)\right) = 1 \qquad \text{by double angle formula}$$

Repeating this process, the product telescopes to:

$$\implies \frac{\sin(\theta)\cos\left(\frac{\theta}{2^{2025}}\right)}{\sin(\theta)\cos(\theta)} = 1$$

$$\implies \frac{\cos\left(\frac{\theta}{2^{2025}}\right)}{\cos(\theta)} = 1$$

$$\implies \cos\left(\frac{\theta}{2^{2025}}\right) = \cos(\theta)$$

$$\implies \frac{\theta}{2^{2025}} = 2n\pi \pm \theta \qquad \text{for some integer n}$$

$$\implies \frac{\theta}{2^{2025}} \pm \theta = 2n\pi$$

$$\implies \theta = \frac{2n\pi}{\frac{1}{2^{2025}} \pm 1}$$

$$\implies \theta = \frac{2^{2026} n\pi}{1 \pm 2^{2025}}$$

$$\implies \theta = \boxed{\frac{2^{2026}\pi}{1 + 2^{2025}}} \qquad \text{since } \theta \in [0, 2\pi]$$

10. For positive integers i, let k_i denote the i'th smallest positive integer that is divisible by 5 or by 7. For example, $k_1 = 5$, $k_2 = 7$, $k_3 = 10$, and $k_{11} = 35$. For positive integers n, define $f(n)$ and $g(n)$ as follows:

$$f(n) = \prod_{i=1}^{660} (n - k_i)$$

and

$$g(n) = \sin\left(\frac{n\pi}{2}\right) + 1$$

If m is chosen uniformly at random from the set of integers between 1 and 2100, inclusive, what is the probability that $f(m)^{g(m)} > 0$?

10. $\frac{83}{140}$

Solution: Listing out the first eleven values of k_i, we find that $f(m)$ has a $\frac{11}{35}$ probability of equaling 0, a $\frac{12}{35}$ probability of being positive, and a $\frac{12}{35}$ probability of being negative. We will do casework on these three cases.

When $f(m) = 0$, $g(m)$ must also equal 0, which happens when $m \equiv 3 \pmod 4$, which has a $\frac{1}{4}$ probability of occurring.

When $f(m) > 0$, $g(m)$ can equal any value.

When $f(m) < 0$, $g(m)$ must equal 0 or 2, which happens when $m \equiv 1, 3 \pmod 4$, which has a $\frac{1}{2}$ probability of occurring.

(Note that the first and third cases depend on the fact that both 5 and 7 are relatively prime to 4 and that 2100 is a multiple of $5 \cdot 7 \cdot 4$, meaning each modular remainder of 4 occurs uniformly among values of m satisfying each case).

Then, the total probability is $(\frac{11}{35})(\frac{1}{4}) + (\frac{12}{35})(1) + (\frac{12}{35})(\frac{1}{2}) = \boxed{\frac{83}{140}}$.