Individual Round

Name: ______________________________

Team Name: ______________________________

Instructions

1. Do not begin until instructed to by the proctor.
2. You will have 60 minutes to solve 10 problems.
3. Your score will be the number of correct answers. There is no penalty for guessing or incorrect answers.
4. No calculators or electronic devices are allowed.
5. All submitted work must be your own. You may not collaborate with anyone else during the individual round.
6. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
7. Do not discuss the problems until all papers have been collected.
8. If you have a question or need to leave the room for any reason, please raise your hand quietly.
9. Good luck!

Acceptable Answers

1. All answers must be simplified as much as reasonably possible. For example, acceptable answers include $\sin(1^\circ)$, $\sqrt{43}$, or π^2. Unacceptable answers include $\sin(30^\circ)$, $\sqrt{64}$, or 3^2.
2. All answers must be exact. For example, π is acceptable, but 3.14 or 22/7 is not.
3. All rational, non-integer numbers must be expressed in reduced form $\pm\frac{p}{q}$, where p and q are relatively prime positive integers and $q \neq 0$. For example, $\frac{2}{3}$ is acceptable, but $\frac{4}{6}$ is not.
4. All radicals must be fully reduced. For example, $\sqrt{24}$ is not acceptable, and should be written as $2\sqrt{6}$. Additionally, rational expressions cannot contain radicals in the denominator. For example, $\frac{1}{\sqrt{2}}$ is not acceptable, and should be written as $\frac{\sqrt{2}}{2}$.
5. Answers should be expressed in base 10 unless otherwise specified.
6. Complex numbers should be expressed in the form $a + bi$, where both a and b are written in a form compliant with the rules above. In particular, no complex denominators are allowed. For example, $\frac{1+2i}{1-2i}$ should be written as $-\frac{3}{5} + \frac{4}{5}i$ or $\frac{-3+4i}{5}$.
7. If a problem asks for all solutions, you may give the answers in any order. However, no credit will be given if any solution is missing or any solution is given but not correct.
8. Angle measurements should be given in radians unless otherwise specified.
9. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

Individual Round

1. In quadrilateral $ABCD$, $AB = AD = 20$, $AC = 25$, and $\angle ABC = \angle ADC = 90^\circ$. What is the length BD?

1. 24

Solution: The length and angle conditions force $\triangle ABC$ and $\triangle ADC$ to be $15-20-25$ right triangles symmetric across AC. Then BD is twice one's height to the hypotenuse, i.e. $2 \cdot \dfrac{15 \cdot 20}{25} = \boxed{24}$.

2. I am thinking of a positive four-digit integer that has no 9s and is a perfect square. If I increase each digit by 1, it would still be a perfect square. What integer am I thinking of?

2. 2025

Solution: If I am thinking of b^2, this gives $b^2 + 1111 = a^2$. By difference of squares, $(a+b)(a-b) = 101 \cdot 11$. Since 101 and 11 are both prime, taking $a + b = 1111, a - b = 1$ gives too large of an answer for b^2, so $a + b = 101, a - b = 11$. Then $b = (101 - 11)/2 = 45$, so $b^2 = \boxed{2025}$.

3. Let $ABCD$ be a square of side length 2025. Let O be the circle of radius 90 concentric with square $ABCD$. Let P be a circle of radius 400 fully contained within the interior of square $ABCD$ and chosen uniformly at random. What is the probability that circles O and P intersect each other?

3. $\frac{4\pi}{25}$ **or** $\frac{1152\pi}{12005}$

Solution:
Intended Solution: The desired probability can be calculated by taking the area of the locus of centers for P that satisfy the condition and dividing by the area of the locus of total possible centers for P. In order for circles O and P to intersect each other, their centers have to be within $90 + 400 = 490$ of each other. Then, the locus of points is a circle of radius 490, which has an area of $490^2\pi$. In order for circle P to be fully contained within the interior of square $ABCD$, the center of P has to be in the interior of $ABCD$ and not within 400 of any edge of $ABCD$. Then, the locus of points is a square of side length $2025 - (2 \cdot 400) = 1225$, which has an area of 1225^2. Then, the desired probability is $\frac{490^2\pi}{1225^2} = \boxed{\dfrac{4\pi}{25}}$.

*On competition day, it was noted that there was an oversight in the intended solution. The two circles do not intersect when one circle is fully contained within the other, so the correct answer should have been $\frac{(490^2-310^2)\pi}{1225^2} = \boxed{\dfrac{1152\pi}{12005}}$. Both answers were accepted as correct answers.

4. How many words of length 8, with or without meaning, can be formed from each of the letters in the word EQUATION exactly once so that the following conditions hold: the longest sequence of consecutive vowels is exactly three and the longest sequence of consecutive consonants is exactly two?

4. 10080

Solution: In the word equation, we see that, in the word EQUATION, we have 5 vowels and 3 consonants, all of which are unique.

We can solve this question by understanding that the unit of 2 consonants mentioned in the conditions above, cannot be immediately adjacent to the other consonant in the word EQUATION.

For the purposes of this question, let C1 be the unit of two consonants, V1 be the unit of 3 vowels, C2 be the 3rd consonant, V2 and V3 be the fourth and fifth vowel respectively.

We know, by looking at the conditions, that C1 and C2 can never be next to each other in any word configuration. This leaves us with a maximum of 3 units in between C1 and C2 which is not possible as that would lead to a violation of the vowel condition mentioned above.

This leaves us with a possibility of either 1 or 2 units in between C1 and C2.

This leaves us with the possible combinations being

$$\_C_1\_C_2\_,\quad \_C_1\_\_C_2,\quad C_1\_C_2\_\_,\quad C_1\_\_C_2\_,\quad \_\_C_1\_C_2$$

In these combinations, the second through fifth combinations are all similar as we cannot place the vowel unit anywhere we want without violating the vowel condition mentioned in the question. Thus, the unit of 3 vowels will be fixed in the blank in between the consonant unit and the third consonant.

This leaves the first combination as being the only combination where the unit of vowels can be placed anywhere in the series of blanks.

The number of word configurations possible in the first combination is:

$$2! \cdot {}^3P_1 \cdot {}^2P_1 \cdot 1 \cdot {}^5P_3 \cdot {}^3P_2 = 6 \cdot 2 \cdot 60 \cdot 6 = 4320$$

Since all other word combinations are mentioned as being similar in nature in that the vowel unit's place is fixed in the single blank, the number of word configurations of one of these combinations is:

$$2! \cdot {}^3P_2 \cdot {}^5P_2 \cdot 2! = 60 \cdot 2 \cdot 2 \cdot 6 = 1440$$

Since there are 4 of these similar combinations, the total number of words in these last 4 combinations are $1440 \cdot 4 = 5760$.

The total number of word combinations possible are 5,760 + 4,320 = 10,080.

Thus, the total number of word combinations possible that satisfy the above conditions is $\boxed{10,080}$.

5. Let $x = \sum_{n=1}^{2025} \frac{2n^2-1}{n^2(n+1)^2}$. What is the value of $\sqrt{x}$?

5. $\frac{2025}{2026}$

Solution: We compute the value of this summation, we first use partial fraction decomposition as follows:

$$\frac{2n^2-1}{n^2(n+1)^2} = \frac{An+B}{n^2} + \frac{Cn+D}{(n+1)^2} = \frac{2n-1}{n^2} + \frac{-2n-1}{(n+1)^2}$$

We now rewrite the sum so that we can apply telescoping in the summation:

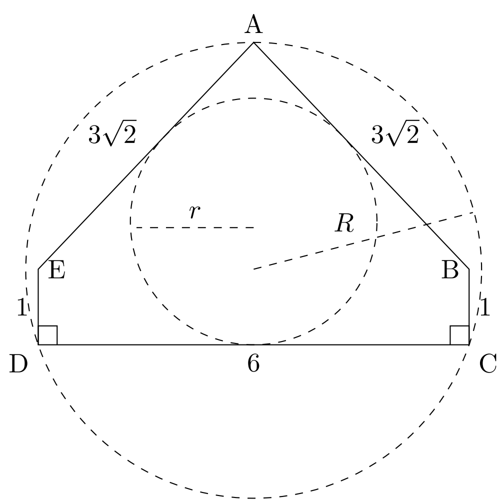
$$\frac{2n-1}{n^2} + \frac{-2n-1}{(n+1)^2} = \frac{2n-1}{n^2} - \frac{2(n+1)-1}{(n+1)^2}$$

$$\sum_{n=1}^{2025} \frac{2n^2-1}{n^2(n+1)^2} = \sum_{n=1}^{2025}\left(\frac{2n-1}{n^2} - \frac{2(n+1)-1}{(n+1)^2}\right) = \frac{2\cdot 1-1}{1^2} - \frac{2(2025+1)-1}{(2025+1)^2}$$

$$= \frac{2026^2 - 2\cdot 2026 + 1}{2026^2} = \left(\frac{2025}{2026}\right)^2$$

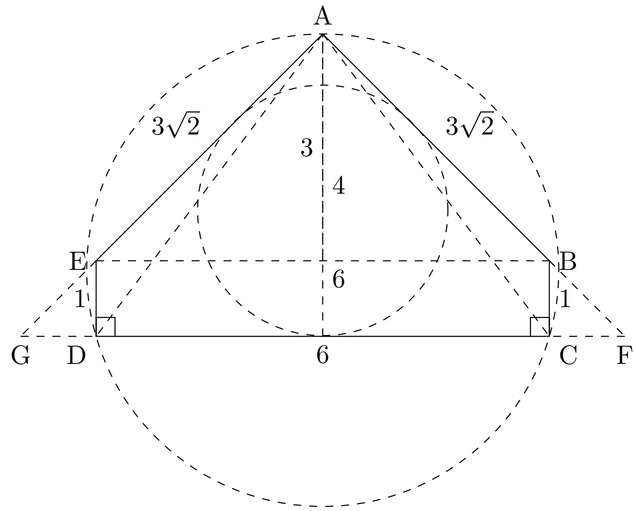
Taking the square root of this expression, we get the answer $\boxed{\frac{2025}{2026}}$.

6. Let $ABCDE$ be a pentagon with side lengths $AB = EA = 3\sqrt{2}$, $BC = DE = 1$, and $CD = 6$. Suppose also that $\angle BCD$ and $\angle CDE$ are right angles. Let r equal the radius of the largest circle that is fully enclosed within $ABCDE$, and let R equal the radius of the smallest circle that fully encloses $ABCDE$. What is $\frac{r}{R}$?



6. $\frac{32(\sqrt{2}-1)}{25}$

Solution: We can extend the line segments $\overline{AE}$, $\overline{AB}$, and $\overline{DC}$ to create $\triangle AFG$ shown in the diagram, as well as drawing $\overline{EB}$. We will also draw lines $\overline{AD}$ and $\overline{AC}$.



Then $\triangle AFG$ is a 45-45-90 triangle and we can find that the altitude from $\overline{EB}$ is 3. Then we find that the altitude of $\triangle AFG$ from $\overline{GF}$ is 4. Since these triangles are similar we can find that $\overline{AG} = 4\sqrt{2}, \overline{AF} = 4\sqrt{2}, \overline{GF} = 8$. Then the smaller circle is the incircle of $\triangle AFG$. We can use the incircle radius formula $r = \frac{2A}{a+b+c}$. Using this we find that the radius of the smaller circle $r = \frac{32}{8+8\sqrt{2}} = \frac{4}{1+\sqrt{2}}$.

Now we will find the radius of the larger circle. This circle is the circumcircle of $\triangle ACD$. $R = \frac{abc}{4A}$. This gets us $R = \frac{150}{48} = \frac{25}{8}$.

Now to find the ratio of the areas we calculate $\frac{r}{R} = \frac{8 \cdot 4 \cdot (\sqrt{2}-1)}{25} = \boxed{\frac{32(\sqrt{2}-1)}{25}}$

7. A line through the point $(c, 2025)$ is tangent to the curve $y = x^4 + 20x^3 + 25$ at two points. What is the value of c?

7. $\frac{9}{2}$

Solution: Let the line be $y = ax + b$. Then $x^4 + 20x^3 + 25 - (ax + b)$ must have two double roots, i.e. it is the square of some polynomial $(x^2 + c_1x + c_2)$ with two real roots. Equating coefficients, $20 = 2c_1$ and $0 = c_1^2 + 2c_2$, so $c_1 = 10$ and $c_2 = -50$. Then $a = -2c_1c_2 = 1000$ and $b = 25 - c_2^2 = -2475$. Solving $ac - b = 2025$, $c = (2025 + 2475)/1000 = \boxed{\frac{9}{2}}$.

8. Suppose $P(x) = ax^2 + bx + c$ is a quadratic polynomial such that

$$\sum_{n=1}^{\infty} \frac{P(n)}{2^n} = \sum_{n=1}^{\infty} \frac{P(n)}{3^n} = 2025.$$

Compute $b + c$.

8. 6075

Solution: Define $S_{k,p}$ as follows:

$$S_{k,p} = \sum_{n=1}^{\infty} \frac{n^p}{k^n}.$$

Note $S_{k,0} = \frac{\frac{1}{k}}{1-\frac{1}{k}} = \frac{1}{k-1}$ by the geometric series formula. Let us compute $S_{k,1}$ in terms of $S_{k,0}$:

$$S_{k,1} = \sum_{n=1}^{\infty} \frac{n}{k^n}$$

$$\implies \frac{1}{k} S_{k,1} = \sum_{n=1}^{\infty} \frac{n}{k^{(n+1)}}$$

$$= \sum_{n=2}^{\infty} \frac{n-1}{k^n}$$

$$\implies S_{k,1} - \frac{1}{k} S_{k,1} = \frac{1}{k} + \sum_{n=2} v^{\infty} \left(\frac{n}{k^n} - \frac{n-1}{k^n} \right)$$

$$= \frac{1}{k} + \sum_{n=2}^{\infty} \frac{1}{k^n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{k^n}$$

$$= S_{k,0}$$

$$\implies \frac{k-1}{k} S_{k,1} = S_{k,0}$$

$$\implies S_{k,1} = \frac{k}{k-1} S_{k,0}$$

Now let us compute $S_{k,2}$ in terms of $S_{k,1}$ and $S_{k,0}$:

$$S_{k,2} = \sum_{n=1}^{\infty} bn^2 k^n$$

$$\implies \frac{1}{k} S_{k,2} = \sum_{n=1}^{\infty} \frac{n^2}{k^{(n+1)}}$$

$$= \sum_{n=2}^{\infty} \frac{(n-1)^2}{k^n}$$

$$\implies S_{k,1} - \frac{1}{k} S_{k,1} = \frac{1}{k} + \sum_{n=2}^{\infty} \left(\frac{n^2}{k^n} - \frac{n^2 - 2n + 1}{k^n} \right)$$

$$= \frac{1}{k} + \sum_{n=2}^{\infty} \frac{2n-1}{k^n}$$

$$= \sum_{n=1}^{\infty} \frac{2n-1}{k^n}$$

$$= 2 \sum_{n=1}^{\infty} \frac{n}{k^n} - \sum_{n=1}^{\infty} \frac{1}{k^n}$$

$$= 2S_{k,1} - S_{k,0}$$

$$\implies \frac{k-1}{k} S_{k,1} = 2S_{k,1} - S_{k,0}$$

$$\implies S_{k,1} = \frac{k}{k-1} (2S_{k,1} - S_{k,0})$$

Then, evaluating on $k = 2$ and $k = 3$, we have $S_{2,0} = 1$, $S_{2,1} = 2$, $S_{2,2} = 6$, $S_{3,0} = \frac{1}{2}$, $S_{2,1} = \frac{3}{4}$, and $S_{2,2} = \frac{3}{2}$.

$$\sum_{n=1}^{\infty} \frac{P(n)}{2^n} = \sum_{n=1}^{\infty} \frac{P(n)}{3^n} = 2025$$

$$\implies \sum_{n=1}^{\infty} \frac{ax^2 + bx + c}{2^n} = \sum_{n=1}^{\infty} \frac{ax^2 + bx + c}{3^n} = 2025$$

$$\implies aS_{2,2} + bS_{2,1} + cS_{2,0} = aS_{3,2} + bS_{3,1} + cS_{3,0} = 2025$$

$$\implies 6a + 2b + c = \frac{3}{2}a + \frac{3}{4}b + \frac{1}{2}c = 2025$$

$$\implies 6a + 2b + c = 2025, 6a + 3b + 2c = 8100$$

$$\implies b + c = 8100 - 2025 = \boxed{6075}$$

9. Consider the sequence of functions $f_0(x), f_1(x), f_2(x), \ldots, f_n(x), \ldots$ where the sequence is defined recursively such that $f_{n+1}(x) = f_n(x) + \sqrt{3} f_n(x - \pi)$. Let $f_0(x) = \cos(\frac{1}{2}x)$, and let k be the smallest positive integer such that $f_{0 \cdot k}(2025), f_{1 \cdot k}(2025), f_{2 \cdot k}(2025), \ldots, f_{i \cdot k}(2025), \ldots$ is a geometric sequence.

What is the common ratio r of this geometric sequence?

9. -8

Solution: When $f_0 = \cos(\frac{1}{2}x)$, $f_0(x-\pi) = \cos(\frac{1}{2}x - \frac{\pi}{2}) = \sin(\frac{1}{2}x)$. Thus, $f_1 = \cos(\frac{1}{2}x) + \sqrt{3}\sin(\frac{1}{2}x)$. Working backwards from the trig identity for $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$, we can deduce $f_1 = 2\cos(\frac{1}{2}x - \frac{\pi}{3})$.

This has a similar form to f_0 in the form $f = a\cos(\frac{1}{2}x - \phi)$. Therefore, let us generalize f_{n+1} for this form. $f(x-\pi) = \cos(\frac{1}{2}x - \phi - \frac{\pi}{2}) = \sin(\frac{1}{2}x - \phi)$. Therefore, $f_{n+1} = \cos(\frac{1}{2}x - \phi) + \sqrt{3}\sin(\frac{1}{2}x - \phi)$. By the same trig identity from above, $f_{n+1} = 2\cos(\frac{1}{2}x - \phi - \frac{\pi}{3}) = 2f(x - \frac{2\pi}{3})$.

From this we develop an explicit formula for $f_n(x) = 2^n\cos(\frac{1}{2}(x - \frac{2\pi}{3}n)) = 2^n\cos(\frac{1}{2}x - \frac{\pi}{3}n)$. $f_n(2025) = 2^n\cos(\frac{2025}{2} - \frac{\pi}{3}n)$. Note 2^n already denotes a geometric sequence, so we just need the lowest positive k such that $\cos(\frac{2025}{2} - \frac{\pi}{3}ik)$ stays constant or switches its sign for increasing i. This can happen if $\frac{\pi}{3}k = \pi$, thus $k = 3$, which will make cosine switch signs since $\cos(\frac{2025}{2}) = -\cos(\frac{2025}{2} - \pi n)$ for odd n or $\cos(\frac{2025}{2} - \pi n)$ for even n. Therefore $r = 2^3(-1) = \boxed{-8}$

10. For positive integers m and n, denote m_n as the number m interpreted as a base n number, where we allow for digits that are greater than or equal to the base. For example, $8642_3 = 8\cdot3^3 + 6\cdot3^2 + 4\cdot3^1 + 2\cdot3^0 = 284$, even though the digits 8, 6 and 4 are not usually allowed in the base 3 number system. Let m be a four digit number of the form $A0B0$, where A and B are integer digits between 0 and 9, inclusive, and let k be a positive integer such that:

$$\sum_{n=2}^{k} m_n = 6292_{10}$$

Compute $m_{10} + k$.

10. 3059

Solution:

$$\begin{aligned}
6292_{10} &= \sum_{n=2}^{k} m_n \\
&= \sum_{n=2}^{k} A0B0_n \\
&= \sum_{n=2}^{k} An^3 + Bn \\
&= A\sum_{n=2}^{k} n^3 + B\sum_{n=2}^{k} n \\
&= A\left[\left(\sum_{n=1}^{k} n^3\right) - 1\right] + B\left[\left(\sum_{n=1}^{k} n\right) - 1\right] \\
&= A\left[\left(\sum_{n=1}^{k} n\right)^2 - 1\right] + B\left[\left(\sum_{n=1}^{k} n\right) - 1\right]
\end{aligned}$$

$$= A\left[\left(\sum_{n=1}^{k} n\right)+1\right]\left[\left(\sum_{n=1}^{k} n\right)-1\right]+B\left[\left(\sum_{n=1}^{k} n\right)-1\right]$$
$$= \left[\left(\sum_{n=1}^{k} n\right)-1\right]\left(A\left[\left(\sum_{n=1}^{k} n\right)+1\right]+B\right)$$

Then, the quantity

$$\left[\left(\sum_{n=1}^{k} n\right)-1\right]$$

is a factor of 6292_{10}, so we need search for factors of 6292_{10} that are one less than a triangular number. Prime factorizing, $6292 = 2^2 \cdot 11^2 \cdot 13$, and listing out integers that are one less than a small triangular number, $0, 2, 5, 9, 14, 20, 27, 35, 44, 54, 65, 77, 90, 104, 119$, we see that the only satisfactory integers are 2 and 44. 2 clearly does not work because it forces A and B to be too large. Then,

$$\left[\left(\sum_{n=1}^{k} n\right)-1\right] = 44_{10}$$

and

$$\left[\left(\sum_{n=1}^{k} n\right)+1\right] = 46_{10}$$
$$\implies 44_{10}\left(46_{10}A+B\right) = 6292_{10}$$
$$\implies \left(46_{10}A+B\right) = 143_{10}$$
$$\implies A=3, B=5$$

since A and B are digits between 0 and 9. Finally, we have

$$\sum_{n=1}^{k} n = 45_{10} \implies k=9$$

Then, the desired answer is $m_{10}+k = 3050+9 = \boxed{3059}$