TEAM ROUND

Names: ____

Team Name: ____

INSTRUCTIONS

1. Do not begin until instructed to by the proctor.
2. You will have 60 minutes to solve 10 problems.
3. Your score will be the number of correct answers. There is no penalty for guessing or incorrect answers.
4. **Only the official team answers will be graded.** If you are submitting the official answer sheet for your team, indicate this by writing "(OFFICIAL)" next to your team name. Do not submit any unofficial answer sheets.
5. No calculators or electronic devices are allowed.
6. All submitted work must be the work of your own team. You may collaborate with your team members, but no one else.
7. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
8. Do not discuss the problems with anyone outside of your team until all papers have been collected.
9. If you have a question or need to leave the room for any reason, please raise your hand quietly.
10. Good luck!

ACCEPTABLE ANSWERS

1. All answers must be simplified as much as reasonably possible. For example, acceptable answers include $\sin(1°)$, $\sqrt{43}$, or π^2. Unacceptable answers include $\sin(30°)$, $\sqrt{64}$, or 3^2.
2. All answers must be exact. For example, π is acceptable, but 3.14 or 22/7 is not.
3. All rational, non-integer numbers must be expressed in reduced form $\pm\frac{p}{q}$, where p and q are relatively prime positive integers and $q \neq 0$. For example, $\frac{2}{3}$ is acceptable, but $\frac{4}{6}$ is not.
4. All radicals must be fully reduced. For example, $\sqrt{24}$ is not acceptable, and should be written as $2\sqrt{6}$. Additionally, rational expressions cannot contain radicals in the denominator. For example, $\frac{1}{\sqrt{2}}$ is not acceptable, and should be written as $\frac{\sqrt{2}}{2}$.
5. Answers should be expressed in base 10 unless otherwise specified.
6. Complex numbers should be expressed in the form $a+bi$, where both a and b are written in a form compliant with the rules above. In particular, no complex denominators are allowed. For example, $\frac{1+2i}{1-2i}$ should be written as $-\frac{3}{5}+\frac{4}{5}i$ or $\frac{-3+4i}{5}$.
7. If a problem asks for all solutions, you may give the answers in any order. However, no credit will be given if any solution is missing or any solution is given but not correct.
8. Angle measurements should be given in radians unless otherwise specified.
9. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

TEAM ROUND

1. (4 points) In a unit square, draw any three lines that intersect with the edges of the square exactly twice. What is the minimum number of regions with positive area that the lines can partition the square into?

1. 2

Solution: Due to the condition of intersecting the edges of the square exactly twice, the first line must partition the square into 2 regions. However, if we choose the second and third lines to be the same line as the first line, we get no new regions. Therefore, the minimum number of regions is $\boxed{2}$.

2. (4 points) A circle centered at O with radius 1 is inscribed inside of rhombus $ABCD$. If $OA = OB$, find the length of the perimeter of $ABCD$.

2. 8

Solution: ABCD is a square with side length $2 \cdot r = 2$. Then, the perimeter is $4 \cdot 2 = 8$.

3. (4 points) Let $f(x) = x^2 + x - 2$. How many real roots (not counting multiplicity) does $f(x)^{2024}$ have?

3. 2

Solution: The discriminant of $f(x)$ is $1^2 - 4(1)(-2) = 9 > 0$, so $f(x)$ has 2 roots. $f(x)^{2024}$ only gains multiplicity on the roots of $f(x)$ and gains no new roots, so the number of roots is still $\boxed{2}$.

4. (5 points) Real numbers x and y satisfy the following equation: $\log_x(125) - \log_y(5) = \log_x(25)\log_y(25)$. Compute the value of $\frac{y^3}{x}$.

4. 625

Solution:

$$\log_x(125) - \log_y(5) = \log_x(25)\log_y(25)$$
$$\implies 3\log_x(5) - \log_y(5) = 4\log_x(5)\log_y(5)$$
$$\implies \frac{3}{\log_5(x)} - \frac{1}{\log_5(y)} = \frac{4}{\log_5(x)\log_5(y)}$$
$$\implies 3\log_5(y) - \log_5(x) = 4$$
$$\implies \log_5\left(\frac{y^3}{x}\right) = 4$$
$$\implies \frac{y^3}{x} = 5^4 = \boxed{625}$$

5. (5 points) The maximal value of $x^3(1-x)^5$ over the range $0 < x < 1$ can be expressed as $2^a 3^b 5^c$ where a, b, c are integers. Find $b + c - a$.

5. **32**

Solution: By AM-GM, we have

$$\frac{x^3(1-x)^5}{3^3 5^5} \le \left(\frac{3\cdot x/3 + 5\cdot(1-x)/5}{8}\right)^8 = \frac{1}{8^8}$$

with equality achieved when $x/3 = (1-x)/5$, or $x = 3/8$. The value we are looking for is $2^{-24}3^3 5^5$, so $b + c - a = \boxed{32}$.

6. (5 points) Calculate the number of ordered pairs of positive integer solutions (a, b) to $3^a - 1 = 2^b$.

6. 2

Solution: Two easy solutions to find are $3^1 - 1 = 2^1$ and $3^2 - 1 = 2^3$. We will show that no other solutions exist. Notice that for $b > 2$ we need $3^a - 1 \equiv (-1)^a - 1 \equiv 0 \mod 4$, so a must be even. Writing $a = 2k$, we have $3^{2k} - 1 = (3^k + 1)(3^k - 1) = 2^b$, but the only powers of 2 that are 2 apart are 2 and 4, which makes $k = 1$. But we have already accounted for that solution, so we are done.

7. (6 points) You are locked in a room with 5 zombies. After each minute, each zombie bites one person besides itself (including other zombies) with uniform probability (i.e. you have a 20% chance of being bitten by each zombie). Find the expected number of minutes before you are bitten.

7. 3125/2101

Solution: The probability of being bitten at minute k is given by: $\left(1 - \left(\frac{4}{5}\right)^5\right)^{k-1} \cdot \left(1 - \left(\frac{4}{5}\right)^5\right)$. Therefore the expected number of minutes before being bitten is given by:

$$\sum_{k=1}^{\infty} k\left(1 - \left(\frac{4}{5}\right)^5\right)^{k-1} \cdot \left(1 - \left(\frac{4}{5}\right)^5\right)$$

This is a double geometric sum, which evaluates to $\frac{3125}{2101}$.

8. (6 points) Let O_n represent the sum of the first n positive odd integers. Let E_n represent the sum of the first n positive even integers. Compute the least positive integer n such that the quantity $(E_n)^2 - (O_n)^2$ is greater than 2024.

8. 10

Solution: Notice that $E_n - O_n = 2 - 1 + 4 - 3 + \ldots + 2n - (2n-1) = n$. Also notice that $E_n + O_n$ equals the sum of the first $2n$ positive integers, which equals $\frac{2n(2n+1)}{2} = 2n^2 + n$. Then, $(E_n)^2 - (O_n)^2 = (E_n - O_n)(E_n + O_n) = n(2n^2 + n) = 2n^3 + n^2$. We notice that when $n = 10$, we have $2n^3 + n^2 = 2000 + 100 = 2100 > 2024$. Checking $n = 9$, we clearly see that the result is not greater than 2024. Then, our final answer is $n = \boxed{10}$.

9. (6 points) Let $P(x)$ be a polynomial such that $P(n) = 2n$ for all positive integers n from 1 to 2024, inclusive. If the degree of $P(x)$ is greater than 1, what is the least degree that $P(x)$ can be so that it can satisfy the above condition?

9. 2024

Solution: $P(x) - 2x = 0$ occurs at $x = 1, 2, \ldots, 2024$, but $P(x) - 2x$ has degree equal to that of $P(x)$, so for it to have 2024 roots it must have degree at least 2024. One $P(x)$ that satisfies this is

$$P(x) = (x-2)(x-4)\cdots(x-4048) + 2x.$$

10. (7 points) Anastasia, Brooke, and Carl are throwing darts onto the xy-plane. Anastasia's dart landed at $(0,0)$, and Brooke's dart landed at $(17,7)$. Carl has yet to throw his dart, but it will land at any point within a circle of radius 27 centered at $(0,37)$ with a uniformly random probability. Find the expected area of the triangle with vertices located where each of the darts landed.

10. $\frac{629}{2}$

Solution: Suppose Carl's dart lands at (a,b). Using the shoelace formula, we can find that the area of the triangle is $|7a - 17b|/2$. The area of the triangle depends linearly on the distance between (a,b) and the line intersecting Anastasia's and Brooke's darts. Because this line does not intersect the circle, the average distance between the line and a uniformly random chosen point of the circle is the same as the distance from the center of the circle to the line. Thus, we can plug in $(0,37)$ for (a,b) to get $\boxed{\frac{629}{2}}$.

Note: The problem originally had a typo with $(7,17)$ as the location of Brooke's dart, which would cause the line in the given solution to intersect the circle, yielding a much more difficult problem to solve.

11. (7 points) Suppose x is a real number that satisfies the equation $\sin(x) + \cos(x) = \frac{1}{5}$. Then, the value of $|\sin(4x)|$ can be expressed as $n(\frac{p}{q})^4$, where p, q, and n are pairwise relatively prime positive integers such that n has no factors that are perfect fourth powers. Compute $p + q + n$.

11. 28

Solution: Let $\sin(x) + \cos(x) = y$. Then,

$$(\sin(x) + \cos(x))^2 = y^2$$

$$\implies \sin^2(x) + \cos^2(x) + 2\sin(x)\cos(x) = y^2$$

$$\implies 2\sin(x)\cos(x) = y^2 - 1$$

$$\implies \sin(2x) = y^2 - 1$$

$$\implies \cos(2x) = \pm\sqrt{1 - (y^2 - 1)^2}$$

$$\implies \cos(2x) = \pm y\sqrt{2 - y^2}$$

$$\implies |\sin(4x)| = |2(y^2 - 1)(y\sqrt{2 - y^2})|$$

Plugging in $y = \frac{1}{5}$, we get $|\sin(4x)| = 21(\frac{2}{5})^4$. So $p + q + n = 2 + 5 + 21 = \boxed{28}$.

12. (7 points) Estimate the number of times the letter 'e' appears in the Wolverine Round. If 'e' appears k times and your guess is g, the number of points you will receive on this question is

$$7 \cdot \min\left(\frac{g}{k}, \frac{k}{g}\right)^2.$$

12. **Depends on input**

Solution: The letter 'e' appears 534 times.

13. (8 points) Let AB be a chord of the unit circle such that the measure of the minor arc $\widehat{AB}$ is 150°, and let CD be a chord of the same circle such that CD is perpendicular with AB, the measure of minor arc $\widehat{CD}$ is 90°, the length of BC is greater than the length of BD, and the length of AC is greater than the length of BC. Let E be the point where AB intersects with CD. Find the area bound by EB, EC, and minor arc $\widehat{BC}$.

13. $\frac{\pi}{12} + \frac{2\sqrt{3}-5}{8}$

Solution: Let minor arc $\widehat{BC}$ have measure 2α. $ABCD$ is a cyclic quadrilateral, and $\angle BAC = \angle CDB = \alpha$. Angle $\angle ABC$ has measure $75° - \alpha$, and $\angle CAD = 45°$. Since $\angle CAD + \angle ABC + \angle ABD = 180°$, we get $\angle ABD = \alpha + 60°$. Then, since edges at E meet at right angles, $\angle EBD + \angle EDB = 90°$ so $\alpha = 15°$ and $\widehat{BC}$ has measure 30°. Observe that EBC is a 30-60-90 triangle, and the length of BC can be found with the law of cosines, as

$$BC^2 = 2(1 - \cos 30°) = 2 - \sqrt{3}$$

and it follows that the area os EBC is $\frac{2\sqrt{3}-3}{8}$. The area bound by BC and arc $\widehat{BC}$ is $\pi/12 - \frac{1}{4}$ by the sine angle formula, so in total the area is

$$\frac{\pi}{12} + \frac{2\sqrt{3}-5}{8}.$$

14. (8 points) Suppose x is a real number that satisfies the equation $8x^6 - 40x^3 + 1 = 0$. Compute $32x^5 + \frac{1}{x^5}$.

14. 464

Solution: $8x^6 - 40x^3 + 1 = 0$
$\implies 8x^3 - 40 + \frac{1}{x^3} = 0$
$\implies 8x^3 + \frac{1}{x^3} = 40$

Let $y = 2x + \frac{1}{x}$.

We have $y^2 = 4x^2 + 4 + \frac{1}{x^2}$
$\implies y^2 - 4 = 4x^2 + \frac{1}{x^2}$
$\implies y(y^2 - 4) = (2x + \frac{1}{x})(4x^2 + \frac{1}{x^2})$
$\implies y^3 - 4y = 8x^3 + 4x + \frac{2}{x} + \frac{1}{x^3}$
$\implies y^3 - 4y - 2y = 8x^3 + 4x + \frac{2}{x} + \frac{1}{x^3} - 2(2x + \frac{1}{x})$
$\implies y^3 - 6y = 8x^3 + \frac{1}{x^3}$
$\implies y^3 - 6y = 40$

$\Longrightarrow y^3 - 6y - 40 = 0$
$\Longrightarrow (y-4)(y^2+4y+10) = 0.$

Thus, $y = 4$ since $y^2 + 4y + 10$ has no real roots.

From before, we already had $y^2 - 4 = 4x^2 + \frac{1}{x^2}$, so plugging in $y = 4$, we get $4x^2 + \frac{1}{x^2} = 12$.

Then, $(8x^3 + \frac{1}{x^3})(4x^2 + \frac{1}{x^2}) = 40 \cdot 12$
$\Longrightarrow 32x^5 + 8x + \frac{4}{x} + \frac{1}{x^5} = 480$
$\Longrightarrow 32x^5 + 4y + \frac{1}{x^5} = 480$
$\Longrightarrow 32x^5 + \frac{1}{x^5} = \boxed{464}$.

15. (8 points) Rowley rolls a fair 6-sided dice until he gets a 6. Given that Rowley did not roll any other even numbers and that he rolled a 1 at least once, what is the expected number of times that Rowley rolled the dice?

15. $7/2$

Solution: We start with the first condition that no other even numbers were rolled. The probability of n dice rolls would then be $\left(\frac{3}{6}\right)^{n-1} \cdot \left(\frac{1}{6}\right)$. In other words, 1, 3, or 5 have to be rolled $n-1$ times (each with probability $\frac{3}{6}$), followed by a 6 (with probability $\frac{1}{6}$).

The second condition is that a 1 was rolled at least once. We can incorporate this probability by taking one minus the probability that the opposite occurs (a 1 is not rolled at all). Then the new probability of n dice rolls is $\left(\frac{3}{6}\right)^{n-1} \cdot \left(1 - \left(\frac{2}{3}\right)^{n-1}\right) \cdot \left(\frac{1}{6}\right)$.

This results in the following expected value for the number of dice rolls:

$$\frac{\sum_{n=2}^{\infty}\left[\left(\frac{3}{6}\right)^{n-1}\left(1-\left(\frac{2}{3}\right)^{n-1}\right)\left(\frac{1}{6}\right)n\right]}{\sum_{n=2}^{\infty}\left[\left(\frac{3}{6}\right)^{n-1}\left(1-\left(\frac{2}{3}\right)^{n-1}\right)\left(\frac{1}{6}\right)\right]}$$

$$= \frac{\sum_{n=2}^{\infty}\left[\left(\left(\frac{1}{2}\right)^{n-1}-\left(\frac{1}{3}\right)^{n-1}\right)n\right]}{\sum_{n=2}^{\infty}\left[\left(\left(\frac{1}{2}\right)^{n-1}-\left(\frac{1}{3}\right)^{n-1}\right)\right]}$$

$$= \frac{\sum_{n=2}^{\infty}\left[\left(\frac{1}{2}\right)^{n-1}n\right]-\sum_{n=2}^{\infty}\left[\left(\frac{1}{3}\right)^{n-1}n\right]}{\sum_{n=2}^{\infty}\left[\left(\frac{1}{2}\right)^{n-1}\right]-\sum_{n=2}^{\infty}\left[\left(\frac{1}{3}\right)^{n-1}\right]}$$

$$= \frac{3-\frac{5}{4}}{1-\frac{1}{2}} = \boxed{\frac{7}{2}}.$$

16. Let $\omega_1, \omega_2, \omega_3$ be the complex-valued solutions to the equation $x^3 - 1 = 0$, and let $f(x) = x^3 + x^2 + 2x$. $f(\omega_1), f(\omega_2), f(\omega_3)$ form the vertices of a triangle in the complex plane. Find the area of that triangle.

16. $9\sqrt{3}/4$

Solution: Without loss of generality let $\omega_1 = 1$. Then ω_2, ω_3 are the solutions to $x^2 + x + 1 = 0$. Notice that

$$f(x) = x((x^2 + x + 1) + 1),$$

so $f(\omega_2) = \omega_2$ and $f(\omega_3) = \omega_3$. $f(\omega_1) = f(1) = 4$, so the area of a triangle with vertices at $(\cos 2\pi/3, \sin 2\pi/3)$, $(\cos 4\pi/3, \sin 4\pi/3)$, and $(4,0)$ is $\boxed{9\sqrt{3}/4}$ by any method of finding the area of a triangle (for instance, the shoelace method).

17. (9 points) Carl the quarter is a magical coin whose probability to land on hands changes based on his previous flip. When Carl lands on heads, his next flip has a $\frac{3}{4}$ probability of landing on heads and a $\frac{1}{4}$ probability of landing on tails. When Carl lands on tails, his next flip has an equal probability of landing on heads or tails. Charley the human flips Carl 2024 times. Given that the first flip landed on tails, compute the expected value of the total number of heads, rounded to the nearest integer.

17. 1348

Solution: Let p_n be the probability that the n'th coin lands on heads. We have $p_1 = 0$, $p_2 = \frac{1}{2}$, and for $n \geq 3$, $p_{n+1} = \frac{3}{4}p_n + \frac{1}{2}(1 - p_n) = \frac{1}{4}p_n + \frac{1}{2}$. Notice that for $n \geq 2$,

$$p_n = \sum_{k=1}^{n-1} \frac{1}{2^{2k-1}}.$$

We can show this is true using induction. The base case $n = 2$ clearly works, and given that the equality is true for m, we have that

$$p_{m+1} = \frac{1}{4}p_m + \frac{1}{2} = \frac{1}{2} + \frac{1}{4}\sum_{k=1}^{m-1} \frac{1}{2^{2k-1}} = \frac{1}{2} + \sum_{k=2}^{m} \frac{1}{2^{2k-1}} = \sum_{k=1}^{m} \frac{1}{2^{2k-1}}.$$

Then, our desired expected value is

$$E = \sum_{n=1}^{2024} p_n = 0 + \sum_{n=2}^{2024}\sum_{k=1}^{n-1} \frac{1}{2^{2k-1}}$$

We have

$$\frac{1}{4}E = \frac{1}{4}\sum_{n=2}^{2024}\sum_{k=1}^{n-1} \frac{1}{2^{2k-1}} = \sum_{n=2}^{2024}\sum_{k=2}^{n} \frac{1}{2^{2k-1}} = \sum_{n=3}^{2025}\sum_{k=2}^{n-1} \frac{1}{2^{2k-1}}$$

Subtracting, we get

$$\begin{aligned}
E - \frac{1}{4}E &= \sum_{n=2}^{2024}\sum_{k=1}^{n-1} \frac{1}{2^{2k-1}} - \sum_{n=3}^{2025}\sum_{k=2}^{n-1} \frac{1}{2^{2k-1}} \\
&= \frac{1}{2} + \sum_{n=3}^{2024}\sum_{k=1}^{n-1} \frac{1}{2^{2k-1}} - \sum_{n=3}^{2024}\sum_{k=2}^{n-1} \frac{1}{2^{2k-1}} - \sum_{k=2}^{2024} \frac{1}{2^{2k-1}} \\
&= \frac{1}{2} + \sum_{n=3}^{2024}\left[\sum_{k=1}^{n-1} \frac{1}{2^{2k-1}} - \sum_{k=2}^{n-1} \frac{1}{2^{2k-1}}\right] - \sum_{k=2}^{2024} \frac{1}{2^{2k-1}} \\
&= \frac{1}{2} + \sum_{n=3}^{2024} \frac{1}{2} - \sum_{k=2}^{2024} \frac{1}{2^{2k-1}} \\
&= \frac{2023}{2} - \sum_{k=2}^{2024} \frac{1}{2^{2k-1}}
\end{aligned}$$

$$= \frac{2023}{2} - \frac{1}{8}\left(\frac{1-(\frac{1}{4})^{2023}}{1-\frac{1}{4}}\right)$$

$$\approx \frac{2023}{2} - \frac{1}{8}\left(\frac{1}{1-\frac{1}{4}}\right)$$

$$= \frac{2023}{2} - \frac{1}{6}$$

Then,

$$\frac{3}{4}E = \frac{2023}{2} - \frac{1}{6} \implies E = \frac{4046}{3} - \frac{2}{9} = 1348 + \frac{4}{9} \approx \boxed{1348}$$

18. (9 points) How many positive integers x are there such that $x \le 100$ and $\gcd(6, x)$ has 2 less factors than x?

18. 26

Solution: Observe that $\gcd(6, x)$ can only have 1, 2, 3, or 4 factors. We casework on this number.

If they only share 1 factor, then $(6, x) = 1$ and x must have 3 factors. This can only happen when x is the square of a prime that isn't 2 or 3, of which there are 2 that are less than 100.

If they share 2 factors, then x is either even and not a multiple of 3 or vice versa, and x must have 4 factors. Then, x must be 2^3, 3^3, or $2p$, $3p$ for $p \ne 2, 3$ a prime. Simple counting will yield 22 choices.

$\gcd(6, x)$ having 3 factors is impossible because it would mean $\gcd(6, x) = p^2$, but no square of a prime divides 6.

$\gcd(6, x)$ having 4 factors means 6 divides x with x having 6 factors. Then, x can only be $2 \cdot 3^2$ or $2^2 \cdot 3$.

In total, there are 26 possible choices of x satisfying the condition.

19. (10 points) Triangle ABC has side lengths $AB = 13$, $BC = 14$, and $CA = 15$. Let D be the foot of the altitude from A to side BC, E be a point on AB, and O be the intersection of CE with AD. If $\frac{AE}{EB} = \frac{2}{3}$, find the sum of the areas of AEO and DCO.

19. 2018/55

Solution: Observe that ADB is a 5-12-13 triangle and ADC is a 9-12-15 triangle, so $BD = 5$ and $DC = 9$. Using Menelaus' Theorem, on A, B, C, O,

$$\frac{14}{9} \cdot \frac{DO}{OA} \cdot \frac{2}{3} = 1$$

so $DO/OA = 27/28$. Since $AD = 12$, we have $DO = \frac{324}{55}$, and this immediately gives us that the area of ODC is $\frac{1458}{55}$. Now, note that $\triangle DEA$ has $2/3$ the area of $\triangle ABD$, and $\triangle EAO$ has $28/55$ the area of $\triangle AED$ so the area of $\triangle AEO$ is

$$30 \cdot \frac{2}{3} \cdot \frac{28}{55} = \frac{560}{55}.$$

Thus the sum of the areas is $\frac{2018}{55}$.

20. (10 points) A biased coin with a probability $\frac{1}{2} - \frac{1}{x}$ of landing heads is flipped 2024 times. If the probability of heads being flipped n times achieves a unique maximum at $n = 999$, what is the range of all possible values of

x? Express your answer as an interval.

20. $(150, 162)$

Solution: Let $p = \frac{1}{2} - \frac{1}{x}$. $\mathbb{P}$(number of heads) is log-convex, so we only need $\mathbb{P}$(999 heads) > $\mathbb{P}$(998 heads), i.e. $\binom{2024}{999} p^{999}(1-p)^{1025} > \binom{2024}{998} p^{998}(1-p)^{1026} \iff 1026p > 999(1-p) \iff p > 999/2025$, along with $\mathbb{P}$(999 heads) > $\mathbb{P}$(1000), i.e. $\binom{2024}{999} p^{999}(1-p)^{1025} > \binom{2024}{1000} p^{1000}(1-p)^{1024} \iff 1000(1-p) > 1025p \iff$ $p < 1000/2025$. Simplifying, $p \in \left(\frac{37}{75}, \frac{40}{81}\right) \implies x \in \boxed{(150, 162)}$.

21. (10 points) A farmer has 123 animals, and they need to be given water. A farmer fills a 1000 liter pool with water, and let's the animals drink from it one at a time, refilling it part way between each animal drinking. The n^{th} animal drinks $\frac{1}{2n}$ of the remaining water. After the n^{th} animal has finished drinking, the farmer adds back $\frac{1}{2n+1}$ of the remaining water. The ratio between the amount of water remaining after all of the animals have finished drinking and the farmer has completed the last partial refill and the amount of water at the start is $\frac{a}{b}$, where a and b are relatively prime positive integers. What is $a+b$?

21. 371

Solution: We can express the actions of the n^{th} animal on the amount of water as multiplying by a factor of $1 - \frac{1}{2n}$. Similarly, for the farmer we get a factor of $1 + \frac{1}{2n+1}$. Writing out the first few terms we get:

$$\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1+\frac{1}{245}\right)\left(1-\frac{1}{246}\right)\left(1+\frac{1}{247}\right)$$

$$\frac{1}{2}\cdot\frac{4}{3}\cdot\frac{3}{4}\ldots\frac{246}{245}\cdot\frac{245}{246}\cdot\frac{248}{247}$$

All of the middle terms cancel, leaving the two end terms:

$$\frac{124}{247}$$

Therefore $a = 124$ and $b = 247$, so $a + b = \boxed{371}$.

22. (11 points) Rob the robot is a machine with the unique ability to generate positive integers. For a given positive integer n, let $R(n)$ denote the probability that Rob chooses n when generating a positive integer. Given that $R(n) = \frac{n}{2^{n+1}}$, let P be the probability that if Rob generates two positive integers a and b, the sum $a+b$ is divisible by four. Find P.

22. 13096/50625

Solution: We will use generating functions. Let $f(x) = \sum_{n=1}^{\infty} R(n)x^n = \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} x^n = \frac{1}{4}x + \frac{2}{8}x^2 + \frac{3}{16}x^3 + \ldots$

Note that the coefficient of each term of $f(x)$ represents the probability that the integer corresponding to the power of x is chosen using $R(n)$; for example, the term $\frac{3}{16}x^3$ indicates that there is a $\frac{3}{16}$ probability that 3 is chosen by $R(n)$.

Note that a property of generating functions is that the coefficient of each term of $(f(x))^2$ represents the probability that the sum of two integers chosen using $R(n)$ is equal to the power of x (you should convince yourself of this using a simpler finite polynomial); for example, if $\frac{2023}{2024}x^{2025}$ is a term in $(f(x))^2$, then there is a $\frac{2023}{2024}$ chance that two integers chosen using $R(n)$ sums to 2025.

Therefore, the probability that the sum $a+b$ is divisible by four is equal to the sum of the coefficients of powers of x in $(f(x))^2$ that are multiples of four. In other words, we need to find the sum of the coefficients of x^4, x^8, x^{12}, etc. in $(f(x))^2$.

Let $u_1 = 1, u_2 = i, u_3 = -1, u_4 = -i$ be the fourth roots of unity. Recall that for any polynomial, $g(x)$, the sum of the coefficients of powers of x that are multiples of four is equal to $\frac{g(u_1)+g(u_2)+g(u_3)+g(u_4)}{4} = \frac{g(1)+g(i)+g(-1)+g(-i)}{4}$. Then, our desired probability is $\frac{(f(1))^2+(f(i))^2+(f(-1))^2+(f(-i))^2}{4}$.

$f(x) = \frac{1}{4}x + \frac{2}{8}x^2 + \frac{3}{16}x^3 + \ldots$

$\implies \frac{xf(x)}{2} = \frac{1}{8}x^2 + \frac{2}{16}x^3 + \frac{3}{32}x^4 + \ldots$

$\implies f(x) - \frac{xf(x)}{2} = \frac{1}{4}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}}x^n = \frac{\frac{1}{4}x}{1-\frac{1}{2}x} = \frac{x}{4-2x}$

$\implies \frac{(2-x)f(x)}{2} = \frac{x}{4-2x}$

$\implies f(x) = \frac{x}{(2-x)^2}$

$\implies (f(1))^2 = (\frac{1}{(2-1)^2})^2 = 1$

$(f(i))^2 = (\frac{i}{(2-i)^2})^2 = \frac{7-24i}{625}$

$(f(-1))^2 = (\frac{-1}{(2+1)^2})^2 = \frac{1}{81}$

$(f(-i))^2 = (\frac{-i}{(2+i)^2})^2 = \frac{7+24i}{625}$

$\implies \frac{(f(1))^2+(f(i))^2+(f(-1))^2+(f(-i))^2}{4} = \boxed{\frac{13096}{50625}}$.

23. (11 points) For positive integers n, define f_n recursively as follows: $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-2} + f_{n-1}$ for $n \geq 1$. Find

$$\sum_{j=1}^{12} \left(\prod_{i=1}^{j} \left[\frac{(f_{i+1})^2}{f_i f_{i+3}} \right] \right).$$

23. 304/305

Solution: Let's begin by unpacking the inner product.

$$\prod_{i=1}^{j} \left[\frac{(f_{i+1})^2}{f_i f_{i+3}} \right]$$

$$= \prod_{i=1}^{j} \left[\frac{f_{i+1} f_{i+2}}{f_i} \cdot \frac{f_{i+1}}{f_{i+2} f_{i+3}} \right]$$

Telescoping, we get

$$= \frac{f_2 f_3}{f_1} \cdot \frac{f_{j+1}}{f_{j+2} f_{j+3}} = 2 \frac{f_{j+1}}{f_{j+2} f_{j+3}}$$

Then, the desired sum becomes

$$\sum_{j=1}^{12}\left(2\frac{f_{j+1}}{f_{j+2}f_{j+3}}\right)$$
$$=2\sum_{j=1}^{12}\left(\frac{f_{j+3}-f_{j+2}}{f_{j+2}f_{j+3}}\right)$$
$$=2\sum_{j=1}^{12}\left(\frac{1}{f_{j+2}}-\frac{1}{f_{j+3}}\right)$$

Telescoping again, we get

$$=2\left(\frac{1}{f_3}-\frac{1}{f_{15}}\right)=2\left(\frac{1}{2}-\frac{1}{610}\right)=1-\frac{1}{305}=\boxed{\frac{304}{305}}$$

24. (11 points) Submit a closed interval that is a subset of $[0,1]$. Let ℓ equal the length of your interval, let n equal the number of other teams that submit an interval that overlaps with your interval, and let N equal the total number of teams competing today. Then, your score on this question will be $11\cdot\ell\cdot(1-\frac{n}{N-1})^2$.

24. **Depends on input**