Tiebreaker Round

Name: ______________________________

Team Name: ______________________________

Instructions

1. Do not begin until instructed to by the proctor.
2. You will have 20 minutes to solve 3 problems.
3. When you would like to submit your answers, please inform your proctor.
4. **Your score will be the number of correct answers, with ties broken by time of submission.**
5. No calculators or electronic devices are allowed.
6. All submitted work must be your own. You may not collaborate with anyone else during the individual round.
7. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
8. Do not discuss the problems until all papers have been collected.
9. If you have a question or need to leave the room for any reason, please raise your hand quietly.
10. Good luck!

Acceptable Answers

1. All answers must be simplified as much as reasonably possible. For example, acceptable answers include $\sin(1°)$, $\sqrt{43}$, or π^2. Unacceptable answers include $\sin(30°)$, $\sqrt{64}$, or 3^2.

2. All answers must be exact. For example, π is acceptable, but 3.14 or 22/7 is not.

3. All rational, non-integer numbers must be expressed in reduced form $\pm\frac{p}{q}$, where p and q are relatively prime positive integers and $q \neq 0$. For example, $\frac{2}{3}$ is acceptable, but $\frac{4}{6}$ is not.

4. All radicals must be fully reduced. For example, $\sqrt{24}$ is not acceptable, and should be written as $2\sqrt{6}$. Additionally, rational expressions cannot contain radicals in the denominator. For example, $\frac{1}{\sqrt{2}}$ is not acceptable, and should be written as $\frac{\sqrt{2}}{2}$.

5. Answers should be expressed in base 10 unless otherwise specified.

6. Complex numbers should be expressed in the form $a + bi$, where both a and b are written in a form compliant with the rules above. In particular, no complex denominators are allowed. For example, $\frac{1+2i}{1-2i}$ should be written as $-\frac{3}{5} + \frac{4}{5}i$ or $\frac{-3+4i}{5}$.

7. If a problem asks for all solutions, you may give the answers in any order. However, no credit will be given if any solution is missing or any solution is given but not correct.

8. Angle measurements should be given in radians unless otherwise specified.

9. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

Tiebreaker Round

1. Suppose a_n is a sequence with $a_1 = 2024$, and for all integers $n \geq 2$, we have that a_n is generated by doubling a_{n-1} and rearranging the resulting digits in increasing order. For instance, since $2 \cdot a_1 = 2 \cdot 2024 = 4048$, we have that $a_2 = 0448 = 448$. Compute a_{2024}.

1. 358

Solution: We will search for a cycle within the sequence. Listing out the first 15 terms, we see that $a_3 = a_{15} = 689$. Then, since each term of the sequence depends only on the previous term, we have $a_m = a_{m+12}$ for all integers $m \geq 3$. Therefore, $a_{2024} = a_{2024 (mod 12)} = a_8 = \boxed{358}$.

2. A license plate number consists of any combination of three letters followed by any combination of three digits. The value of a license plate is determined by adding the value of each character in the license plate, with letters in the first half of the alphabet being worth 4 points, letters in the second half of the alphabet being worth 2 points, and each digit being worth its corresponding value. There are $a \cdot 13^b$ license plates with a value of 30, where a and b are non-negative integers such that $\gcd(a, 13) = 1$. What is $a + b$?

2. 239

Solution: The total value of the letters is either $6, 8, 10$, or 12, with there being $13^3, 3 \cdot 13^3, 3 \cdot 13^3, 13^3$ possible combinations, respectively. For the value of the numbers, note that for all k such that $18 \leq k \leq 27$, the total number of combinations of three digits totalling k is given by the $(k+1)^{\text{th}}$ triangular number, i.e. $\frac{(k+1)(k+2)}{2}$. Therefore, the total number of combinations can be written as $(10 + 3 \cdot 21 + 3 \cdot 36 + 55) \cdot 13^3 = 236 \cdot 13^3$. Therefore, the answer is $236 + 3 = \boxed{239}$.

3. Let $f(n, b)$ be a function that expresses a positive integer $n > 2$ in base $b > 1$ and then returns that expression as if it was written in base $n + 1$. For example, $4 = 100_2$ so $f(4, 2) = 100_5 = 25$. We say a number n is *optimal* if n does not divide $f(n, b)$ for all integers b such that $1 < b < n$. What is the sum of the smallest 10 optimal numbers that are not prime?

3. 112

Solution: We first express n as a sum of powers of b: $n = \sum_k a_k b^k$. We then have that $f(n, b) = \sum_k a_k (n+1)^k$. Expanding this sum using the binomial theorem, each term is divisible by n except for the following sum: $\sum_k a_k$, which is clearly less than n because $b > 1$. Therefore $\sum_k a_k$ is not divisible by n for any valid choice of n and b. We then have that all $n > 2$ are optimal, so the sum of the first 10 optimal numbers that aren't prime is given by $4 + 6 + 8 + 9 + 10 + 12 + 14 + 15 + 16 + 18 = \boxed{112}$.