Team Round

Names: ______________________________

Team Name: ______________________________

Instructions

1. Do not begin until instructed to by the proctor.
2. You will have 60 minutes to solve 10 problems.
3. Your score will be the number of correct answers. There is no penalty for guessing or incorrect answers.
4. **Only the official team answers will be graded.** If you are submitting the official answer sheet for your team, indicate this by writing "(OFFICIAL)" next to your team name. Do not submit any unofficial answer sheets.
5. No calculators or electronic devices are allowed.
6. All submitted work must be the work of your own team. You may collaborate with your team members, but no one else.
7. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
8. Do not discuss the problems with anyone outside of your team until all papers have been collected.
9. If you have a question or need to leave the room for any reason, please raise your hand quietly.
10. Good luck!

Acceptable Answers

1. All answers must be simplified as much as reasonably possible. For example, acceptable answers include $\sin(1°)$, $\sqrt{43}$, or π^2. Unacceptable answers include $\sin(30°)$, $\sqrt{64}$, or 3^2.

2. All answers must be exact. For example, π is acceptable, but 3.14 or 22/7 is not.

3. All rational, non-integer numbers must be expressed in reduced form $\pm\frac{p}{q}$, where p and q are relatively prime positive integers and $q \neq 0$. For example, $\frac{2}{3}$ is acceptable, but $\frac{4}{6}$ is not.

4. All radicals must be fully reduced. For example, $\sqrt{24}$ is not acceptable, and should be written as $2\sqrt{6}$. Additionally, rational expressions cannot contain radicals in the denominator. For example, $\frac{1}{\sqrt{2}}$ is not acceptable, and should be written as $\frac{\sqrt{2}}{2}$.

5. Answers should be expressed in base 10 unless otherwise specified.

6. Complex numbers should be expressed in the form $a + bi$, where both a and b are written in a form compliant with the rules above. In particular, no complex denominators are allowed. For example, $\frac{1+2i}{1-2i}$ should be written as $-\frac{3}{5} + \frac{4}{5}i$ or $\frac{-3+4i}{5}$.

7. If a problem asks for all solutions, you may give the answers in any order. However, no credit will be given if any solution is missing or any solution is given but not correct.

8. Angle measurements should be given in radians unless otherwise specified.

9. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

Team Round

1. You have 5 segments of lengths 1, 2, 4, 4, 5. The perimeter of a triangle is built from all of the segments, where each segment is a part of exactly one side of the triangle. What is the largest possible area of such a triangle?

1. 12

Solution: There are only 4 possible triangles one can form with these segments:

$$A = 4, B = 5, C = 1 + 2 + 4 = 7$$
$$A = 4, B = 1 + 5 = 6, C = 2 + 4 = 6$$
$$A = 4, B = 2 + 5 = 7, C = 1 + 4 = 5$$
$$A = 1 + 4 = 5, B = 2 + 4 = 6, C = 5$$

After applying Heron's formula to each of the triangle we found that the largest triangle one can have is: $A = \sqrt{s(s-5)(s-5)(s-6)}$, where $s = (5 + 5 + 6)/2 = 8$, so $\boxed{\text{A = 12}}$.

2. You have two fair 6-sided dice and roll them 2 times. Find the probability of getting a sum of 6 exactly once.

2. **155/648**

Solution: In the 36 number of total combined outcomes of the two dice throw, there are 5 outcomes where the two dice values sum up to 6: $(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)$. So the probability of succeeding exactly 1 times out of 2 is $\left(\frac{5}{36}\right)\left(\frac{31}{36}\right)$. The number of outcomes which in total 1 throws succeeded and 1 throws do not is $\frac{2!}{1!1!}$. So the total probability is $P(5) = \left(\frac{5}{36}\right)\left(\frac{31}{36}\right)\frac{2!}{1!1!} = 155/648$.

3. Find the last two digits of $1^{40} + 3^{40} + 5^{40} + \cdots + 37^{40} + 39^{40}$.

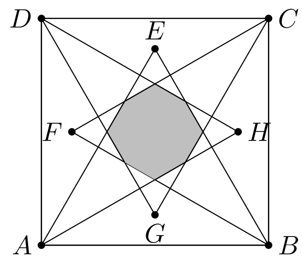
3. **16**

Solution: Since $\varphi(100) = 40$, any odd number not divisible by 5 contributes 1 to the sum by Euler's theorem. Thus, we only have to worry about $5, 15, 25$, and 35. But $5^{40}n^{40} \equiv 5^{40} \cdot n^{40} \equiv 5^{40} \cdot 1 \mod 100$ when 5 does not divide n, so our list is narrowed to just 5 and 25. Since $5^2 \equiv 5^3 \equiv 5^4 \mod 100$,

$$5^{40} \equiv 5^{20} \equiv 5^{10} \equiv 25 \cdot 5^8 \equiv 5^6 \equiv 5^4 \equiv 25 \mod 100.$$

We also have $25^{40} \equiv (5^{40})^2 \equiv 25^2 \equiv 25 \mod 100$, so the sum modulo 100 is $4 \cdot 25 + 16 \equiv \boxed{16} \mod 100$.

4. Let $ABCD$ be a square of side length 1. Construct four equilateral triangles ABE, BCF, CDG, and DAH such that points E, F, G, H all lie within $ABCD$. Find the area of the intersection of all four triangles.



4. $3 - \frac{5\sqrt{3}}{3}$

Solution: To simplify the problem, we will only consider the area of the intersection within the 45-90-45 triangle with vertices A, the center of $ABCD$ (call this point O), and the midpoint of AB (call this point I). Additionally, call the intersection of $\overline{AH}$ and $\overline{BF}$ point J and call the intersection of $\overline{BF}$ and $\overline{DG}$ point K. We have the following expression for the area of the shaded region: Area($\triangle AOI$) − Area($\triangle AIJ$) − Area($\triangle AKJ$). Trivially, Area($\triangle AOI$) $= \frac{1}{8}$. Knowing that $\triangle AIJ$ is a right triangle with $\overline{AI} = \frac{1}{2}$ and $m\angle IAJ = 30°$, we get that Area($\triangle AIJ$) $= \frac{1}{8\sqrt{3}}$. From this, we also get that $\overline{AJ} = \frac{1}{\sqrt{3}}$. It is also easy to see that $m\angle JAK = 15°$. Using coordinate geometry, we get that $\overline{AK} = \frac{\sqrt{2}}{1+\sqrt{3}}$. Using the half-angle formula to find $\sin(15°)$, and the SAS area formula, we get that Area($\triangle AKJ$) $= \frac{1}{2\sqrt{3}} - \frac{1}{4}$. Plugging these back in, multiplying by 8, and rationalizing, we get the area of the entire shaded region to be $\boxed{3 - \frac{5}{3}\sqrt{3}}$.

5. Let $f(x) = 1 + 4x + 9x^2 + \cdots = \sum_{n=0}^{\infty}(n+1)^2x^n$. The value of $f(\frac{1}{11})$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. What is $m+n$?

5. **613**

Solution: Knowing that $1 + x + x^2 + \ldots$ can easily be written as $\frac{1}{1-x}$, we will manipulate $f(x)$ to get a geometric series. We will also use summation notation.

$$f(x) = \sum_{k=0}^{\infty}(k+1)^2x^k$$

$$f(x) - xf(x) = \sum_{k=0}^{\infty}(k+1)^2x^k - \sum_{k=0}^{\infty}(k+1)^2x^{k+1}$$

$$f(x)(1-x) = 1 + \sum_{k=1}^{\infty}(k+1)^2x^k - \sum_{k=1}^{\infty}k^2x^k$$

$$f(x)(1-x) = \sum_{k=0}^{\infty}(2k+1)x^k$$

$$f(x)(1-x) - x\big(f(x) - xf(x)\big) = \sum_{k=0}^{\infty}(2k+1)x^k - \sum_{k=0}^{\infty}(2k+1)x^{k+1}$$

$$f(x)(1-x) - x\big(f(x) - xf(x)\big) = 1 + \sum_{k=1}^{\infty}(2k+1)x^k - \sum_{k=1}^{\infty}(2k-1)x^k$$

$$f(x)(x-1)^2 = 1 + 2\sum_{k=1}^{\infty} x^k$$

$$f(x)(x-1)^2 = -1 + 2\sum_{k=0}^{\infty} x^k$$

$$f(x)(x-1)^2 = -1 + \frac{2}{1-x}$$

$$f(x) = -\frac{1+x}{(x-1)^3}$$

Plugging in $x = \frac{1}{11}$ gives $f(x) = \frac{363}{250}$, so $m + n = \boxed{613}$.

6. The number of arithmetic sequences with only integer terms such that the starting term is 0 and the ending term is 2024^{2024} can be expressed as pn^2 where n is a positive integer and p is prime. What is $p + n$?

6. **8098**

Solution: Because all of the arithmetic sequences we need to count start and end on the same numbers, the only distinction between sequences are the differences between consecutive terms. The difference between any two terms in an arithmetic sequence must be divisible by the difference between two consecutive terms, so we have that the difference d between consecutive terms of any of the sequences must divide 2024^{2024}. Factoring: $2024 = 2^3 \cdot 11 \cdot 23$, so $2024^{2024} = 2^{6072} \cdot 11^{2024} \cdot 23^{2024}$. Then, we have that the number of divisors of 2024^{2024} is $6073 \cdot 2025 \cdot 2025$. Clearly, the two factors of 2025 are part of n^2. If 6073 is not prime than it must be of the form qm^2 where q is prime and m is an integer. So we only have to check for prime factors up to $6073^{\frac{1}{3}} \approx 18$. After checking for divisibility by $2, 3, 5, 7, 11, 13$, and 17, we conclude that 6073 is prime. Thus $p = 6073$ and $n = 2025$, so $p + n = \boxed{8098}$.

7. Let f and g such that for all $x, y \in \mathbb{R}$, $f(xy)g(x) = f(x) + g(y)$ and $f(0), g(0) > 0$. What is the value of the expression $\frac{f(2)}{g(4)} - f(6)$?

7. **-1**

Solution: Plugging in $x = 0$, we get that $g(y) = f(0)(g(0) - 1)$ for all $y \in \mathbb{R}$, i.e. g is a constant function. Plugging in $y = 0$ into the original expression, we get that $f(x) = f(0)g(x) - g(0)$. Plugging in the previous expression found for an arbitrary input for g, we get $f(x) = f(0)(g(0) - 1)(f(0) - 1)$ for all $x \in \mathbb{R}$, which is also a constant function. Setting $x = 0$ and dividing by $f(0) \neq 0$, we get that $1 = (g(0) - 1)(f(0) - 1)$. Define $f(x) := H + 1$. Plugging this into the previous expression we get $g(0) = g(x) = \frac{H+1}{H}$. Finally, we get that $\frac{f(2)}{g(4)} - f(6) = \frac{H+1}{\frac{H+1}{H}} - (H + 1) = \boxed{-1}$.

8. What is the smallest positive integer a such that 5^{2024} divides $2^a + 3^a$? You may express your answer as a product of powers of primes.

8. 5^{2023}

Solution: Note that 2^a cycles as $1, 2, 4, 3, 1, \ldots \mod 5$ and 3^a cycles as $1, 3, 4, 2, 1 \ldots \mod 5$, so $2^a + 3^a$ is only divisible by 5 when $a \equiv 1, 3 \mod 4$, or when a is odd. Let $v_p(n)$ denote the p-adic evalulation of n. Since a is odd and $5 \mid 2 + 3$, by the LTE lemma

$$v_5(2^a + 3^a) = v_5(2 + 3) + v_5(a) = 1 + v_5(a) \geq 2024,$$

so we need $v_5(a) \geq 2023$. Thus, $a = \boxed{5^{2023}}$ is the smallest possible solution.

9. How many permutations $f : \{1, 2, 3, 4, 5, 6\} \to \{1, 2, 3, 4, 5, 6\}$ are there such that f is the square of another permutation, i.e. there exists a permutation of g of $\{1, 2, 3, 4, 5, 6\}$ such that for all $1 \leq k \leq 6$, $f(k) = g(g(k))$?

9. **270**

Solution: For any permutation f we can generate its functional graph G_f by letting $\{1, 2, 3, 4, 5, 6\}$ be vertices and adding a directed edge $i \to j$ if $f(i) = j$. Since f is bijective, G_f is only composed of cycles. We consider even and odd length cycles of f^2: an odd-length cycle $i_1 \to i_2 \to \cdots \to i_{2k+1} \to i_1$ turns into

$$i_1 \to i_3 \to \cdots \to i_{2k+1} \to i_2 \to i_4 \to i_{2k} \to i_0$$

by moving twice along the cycle. The image is a single cycle of the same length with the same elements; this forms a bijection between odd-length cycles in f and odd length cycles in f^2. An even-length cycle $i_1 \to i_2 \to \cdots \to i_{2k}$ turns into two cycles

$$i_1 \to i_3 \to \cdots \to i_{2k-1} \to i_1$$

$$i_2 \to i_4 \to \cdots \to i_{2k} \to i_2.$$

In particular, when k is even, a $2k$-length cycle must split into two k-length cycles. Since this is the only way to generate even-length cycles in f^2, we can characterize squares of permutations as only those having an even number of cycles of even length in its functional graph. Given a graph with this property we can find its "square root" by using the bijection with odd-length cycles and combining even-length cycles with equal length by alternating elements.

For six elements, the cycle lengths of graphs satisfying the above property must be one of $(1, 1, 1, 1, 1, 1)$, $(2, 2, 1, 1)$, $(3, 1, 1, 1)$, $(3, 3)$, or $(5, 1)$ where a cycle of length $k > 2$ has $(k-1)!$ configurations with the same elements corresponding to different permutations of these integers. So the final count is

$$1 + \frac{\binom{6}{2}\binom{4}{2}}{2} + \binom{6}{3} \cdot 2 + \frac{\binom{6}{3}\binom{3}{3}}{2} \cdot 4 + \binom{6}{5} \cdot 4! = \boxed{270}.$$

10. A positive integer is considered *cascading* if its digits when expressed in base 10 are strictly decreasing from left to right. For example, 8420, 763, and 5 are cascading numbers, but 8843 and 123 are not.

What is the sum of all cascading numbers? You may express your answer in base 11, where the letter A represents the base 11 digit for the decimal number 10.

10. $\underline{A801548430_{11}} = 25296994503$

Solution: Consider restricting which digits we use to generate our list of cascading numbers. Let $C(n)$ be the set of cascading numbers with smallest digit at least n, and let $S(n)$ be the sum of all elements of $C(n)$. Trivially, $C(9) = \{9\}$. We also get that $C(8) = \{9, 98, 8\}$ and $C(7) = \{9, 98, 8, 97, 987, 87, 7\}$. Note that for each digit we add to our set, we get three subsets of cascading numbers. First, there is the previous set of cascading numbers. Second, there is a copy of the previous set with the new digit appended to the end of each cascading number, which effectively multiplies the sum by 10 and adds the new digit multiplied by the number of elements in the previous set. Lastly, there is the new digit. Therefore, $S(n-1) = 11S(n) + k(n-1)$, where k is the number of elements in $C(n)$ plus 1. Using induction, $k = 2^{9-n}$. By applying this formula iteratively, we get that $C(0) = 11^9 \cdot 9 + 11^8 \cdot 2 \cdot 8 + 11^7 \cdot 2^2 \cdot 7 + 11^6 \cdot 2^3 \cdot 6 + 11^5 \cdot 2^4 \cdot 5 + 11^4 \cdot 2^5 \cdot 4 + 11^3 \cdot 2^6 \cdot 3 + 11^2 \cdot 2^7 \cdot 2 + 11^1 \cdot 2^8 \cdot 1$. Writing this in a more compact form that allows for digit values greater than A (10 in base 11): $[9, 16, 28, 48, 80, 128, 192, 256, 256, 0]_{11}$. After some calculation, this becomes $[10, 8, 0, 1, 5, 4, 8, 4, 3, 0]_{11}$. Therefore, the answer is $\boxed{A801548430_{11}}$.