Proposition 2: Team Round

Names: ____________________

Team ID: ____________________

Instructions

1. Do not begin until instructed to by the proctor.
2. You will have 30 minutes to solve 10 problems.
3. Your score will be the number of correct answers. There is no penalty for guessing or incorrect answers.
4. **Only the official team answers will be graded.** If you are submitting the official answer sheet for your team, indicate this by writing "(OFFICIAL)" next to your team name. Do not submit any unofficial answer sheets.
5. No calculators or electronic devices are allowed.
6. All submitted work must be the work of your own team. You may collaborate with your team members, but no one else.
7. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
8. Do not discuss the problems with anyone outside of your team until all papers have been collected.
9. If you have a question or need to leave the room for any reason, please raise your hand quietly.
10. Good luck!

Acceptable Answers

1. All answers must be simplified as much as reasonably possible. For example, acceptable answers include $\sin(1°)$, $\sqrt{43}$, or π^2. Unacceptable answers include $\sin(30°)$, $\sqrt{64}$, or 3^2.
2. All answers must be exact. For example, π is acceptable, but 3.14 or 22/7 is not.
3. All rational, non-integer numbers must be expressed in reduced form $\pm\frac{p}{q}$, where p and q are relatively prime positive integers and $q \neq 0$. For example, $\frac{2}{3}$ is acceptable, but $\frac{4}{6}$ is not.
4. All radicals must be fully reduced. For example, $\sqrt{24}$ is not acceptable, and should be written as $2\sqrt{6}$. Additionally, rational expressions cannot contain radicals in the denominator. For example, $\frac{1}{\sqrt{2}}$ is not acceptable, and should be written as $\frac{\sqrt{2}}{2}$.
5. Answers should be expressed in base 10 unless otherwise specified.
6. Complex numbers should be expressed in the form $a + bi$, where both a and b are written in a form compliant with the rules above. In particular, no complex denominators are allowed. For example, $\frac{1+2i}{1-2i}$ should be written as $-\frac{3}{5} + \frac{4}{5}i$ or $\frac{-3+4i}{5}$.
7. If a problem asks for all solutions, you may give the answers in any order. However, no credit will be given if any solution is missing or any solution is given but not correct.
8. Angle measurements should be given in radians unless otherwise specified.
9. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

Proposition 2: Team Round

1. Stephen has an unfair coin where both the probability of flipping heads and the probability of flipping tails is nonzero. Stephen flips the unfair coin three times. The probability that Stephen flips three heads is equal to the probability that Stephen flips two heads and one tails, in any order. Determine the probability that in three flips, Stephen flips three tails.

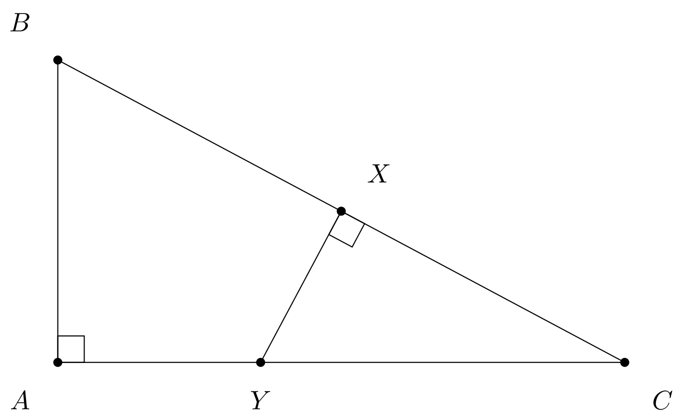
1. $\frac{1}{64}$

Solution: Let p be the probability that the coin is flipped heads, and let $q = 1-p$ be the probability it is flipped tails. Then we know that $p^3 = \binom{3}{2}p^2q$. Since $p \neq 0$, we have $1 - q = p = 3q$, so $q = \frac{1}{4}$. The answer to the problem is thus $q^3 = \frac{1}{64}$.

2. Triangle $\triangle ABC$ has sides $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ of length 8, 15, and 17, respectively. A point P is randomly chosen inside of $\triangle ABC$. What is the probability that $\overline{BP}$ is longer than $\overline{CP}$?

2. $\frac{611}{900}$

Solution: We make the following diagram:



Here, X is the midpoint of $\overline{BC}$. If we pick P randomly in $\triangle ABC$, then the probability that $\overline{BP}$ is longer than $\overline{CP}$ is the area of the quadrilateral $ABXY$ divided by the area of $\triangle ABC$. To find the area of $ABXY$, we will find the areas of $\triangle ABC$ and $\triangle XYC$.

Begin by noting that $\triangle XYC$ is similar to $\triangle ABC$. Since $\overline{XC}$ is of length $\frac{17}{2}$ and $\overline{XC}$ is of length 15, the scaling factor is $\frac{17}{30}$. It follows that the area of $\triangle XYC$ is $\left(\frac{17}{30}\right)^2 = \frac{289}{900}$ times the area of $\triangle ABC$. Thus, the area of $ABXY$ is $1 - \frac{289}{900} = \frac{611}{900}$ times the area of $\triangle ABC$. Finally, the answer must be $\frac{611}{900}$.

3. We can write $\sqrt{20+2\sqrt{91}}+\sqrt{20-2\sqrt{91}}$ in the form $x+\sqrt{y}$, where x and y are integers. Determine $x+y$.

3. 52

Solution: By squaring, we have

$$\begin{aligned}
\sqrt{20+2\sqrt{91}}+\sqrt{20-2\sqrt{91}} &= \sqrt{\left(\sqrt{20+2\sqrt{91}}+\sqrt{20-2\sqrt{91}}\right)^2}\\
&= \sqrt{\left(20+2\sqrt{91}\right)+2\sqrt{\left(20+2\sqrt{91}\right)\left(20-2\sqrt{91}\right)}+\left(20-2\sqrt{91}\right)}\\
&= \sqrt{40+2\sqrt{\left(20+2\sqrt{91}\right)\left(20-2\sqrt{91}\right)}}\\
&= \sqrt{40+2\sqrt{400-364}}\\
&= \sqrt{40+2\cdot 12}\\
&= \sqrt{52}.
\end{aligned}$$

Thus, the answer is 52.

Alternative Solution: By noticing a hidden square of sum and square of difference, we can simplify as follows:

$$\begin{aligned}
\sqrt{20+2\sqrt{91}}+\sqrt{20-2\sqrt{91}} &= \sqrt{\left(\sqrt{13}\right)^2+2\left(\sqrt{13}\right)\left(\sqrt{7}\right)+\left(\sqrt{7}\right)^2}\\
&\quad+\sqrt{\left(\sqrt{13}\right)^2-2\left(\sqrt{13}\right)\left(\sqrt{7}\right)+\left(\sqrt{7}\right)^2}\\
&= \sqrt{\left(\sqrt{13}+\sqrt{7}\right)^2}+\sqrt{\left(\sqrt{13}-\sqrt{7}\right)^2}\\
&= \left(\sqrt{13}+\sqrt{7}\right)+\left(\sqrt{13}-\sqrt{7}\right)\\
&= 2\sqrt{13}\\
&= \sqrt{52}.
\end{aligned}$$

This again yields the solution 52.

4. Consider a regular 2020-gon. Pick three of its vertices A, B, and C at random, without replacement. What is the probability that there is a fourth vertex D such that the points A, B, C, and D are the corners of some rectangle?

4. $\frac{3}{2019}$

Solution: Note that we can pick a vertex D to complete a rectangle if and only if one of $\angle BAC$, $\angle ABC$, or $\angle ACB$ is a right angle. Since our 2020-gon is regular, its vertices lie on a circle. So, one of these angles is a right angle if and only if two of A, B, and C form a diameter of the circle.

The probability that A and B form a diameter is $\frac{1}{2019}$ (think about choosing A first. Then there is only 1 out of 2019 remaining choices for vertices that is directly across from A). Similarly, the probability that A and C form a diameter and the probability that B and C form a diameter are both $\frac{1}{2019}$. These events are disjoint, so the answer to our problem is their sum: $\frac{3}{2019}$.

5. The slope of the common tangent line(s) to the parabolas $y = -x^2 + 4x - 3$ and $y = x^2 - 2x + 3$ are expressible in the form $m = a \pm \sqrt{b}$. Determine $a + b$.

5. 4

Solution: Suppose $y = mx + b$ is a tangent line to both $y = -x^2 + 4x - 3$ and $y = x^2 - 2x + 3$. Considering the first parabola, we must have that $mx + b = y = -x^2 + 4x - 3$ has only one solution. Rewriting, we have that $x^2 + (m - 4)x + (b + 3)$ has one solution. Considering the discriminant, we have $(m - 4)^2 = 4(b + 3)$.

We similarly can find that $(-m - 2)^2 = 4(-b + 3)$ by considering the other parabola.

Adding the two equations, we see $(m - 4)^2 + (-m - 2)^2 = 24$, which is the same as $m^2 - 2m - 2 = 0$. Solving with your favorite method, we see $m = 1 \pm \sqrt{3}$, making the answer 4.

6. Determine the number of ordered triples of sets (A, B, C) where the sets A, B, C satisfy

$$A \subseteq B \subseteq C \subseteq \{1, 2, \ldots, 7\}.$$

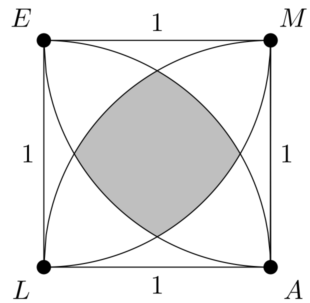
6. 16384

Solution: Consider a single element $n \in \{1, 2, \ldots, 7\}$. There are four cases:

- $n \notin C$,
- $n \in C$, but $n \notin B$,
- $n \in B$, but $n \notin A$,
- $n \in A$.

Furthermore, for $m \in \{1, 2, \ldots, 7\}$ with $m \neq n$, n's case is independent of m's case. This means there are $4^7 = 16384$ total valid choices for $A \subseteq B \subseteq C \subseteq \{1, 2, \ldots, 7\}$.

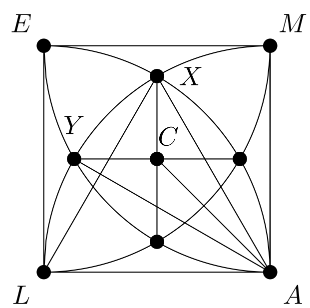
7. Consider the following square $LEMA$:



Each of the arcs is a quarter circle centered at one of L, E, M, and A. What is the area of the shaded region?

7. $\frac{\pi}{3} + 1 - \sqrt{3}$

Solution: We add some auxiliary lines to our diagram as follows:



We are then interested in finding the area of the shape subtended by X, Y, and C, as the shaded region is exactly 4 times as large.

We start by noting that $\angle LAX$ is of measure $\frac{\pi}{3}$ (this follows from the fact that $\triangle LAX$ is equilateral). So $\angle XAM$ is of measure $\frac{\pi}{6}$.

Similarly, $\angle LAY$ is of measure $\frac{\pi}{6}$, which means $\angle YAX$ is of measure $\frac{\pi}{6}$. It follows that the sector subtended by Y and X, and centered at A is $\frac{\pi}{12}$.

Next, we see that Y is a distance of $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ away from $\overline{MA}$; C is a distance of $\frac{1}{2}$ away from $\overline{MA}$. Since both have the same height, and since $\overline{MA}$ is vertical, it follows that $\overline{YC}$ is of length $\frac{-1+\sqrt{3}}{2}$. The area of $\triangle YCA$ is then $\frac{-1+\sqrt{3}}{8}$. The area of $\triangle XCA$ is the same.

Finally, we combine our results. The area of the shape subtended by X, Y, and C is

$$\frac{\pi}{12} - 2 \cdot \frac{-1+\sqrt{3}}{8} = \frac{\pi}{12} - \frac{-1+\sqrt{3}}{4}.$$

So our final answer is

$$4\left(\frac{\pi}{12} - \frac{-1+\sqrt{3}}{4}\right) = \frac{\pi}{3} + 1 - \sqrt{3}.$$

8. Compute
$$\sec\left(\frac{\pi}{9}\right)\sec\left(\frac{2\pi}{9}\right)\sec\left(\frac{4\pi}{9}\right).$$

8. 8

Solution: Let's instead look at $\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right)$.
First, note that

$$\begin{aligned}\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta),\\ \cos(\alpha-\beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta).\end{aligned}$$

Adding, we have

$$\begin{aligned}\cos(\alpha+\beta)+\cos(\alpha-\beta) &= 2\cos(\alpha)\cos(\beta)\\ \frac{1}{2}\left(\cos(\alpha+\beta)+\cos(\alpha-\beta)\right) &= \cos(\alpha)\cos(\beta).\end{aligned}$$

Then, we have

$$\begin{aligned}\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right) &= \frac{1}{2}\left(\cos\left(\frac{\pi}{9}+\frac{2\pi}{9}\right)+\cos\left(\frac{\pi}{9}-\frac{2\pi}{9}\right)\right)\cos\left(\frac{4\pi}{9}\right)\\ &= \frac{1}{2}\left(\frac{1}{2}+\cos\left(\frac{\pi}{9}\right)\right)\cos\left(\frac{4\pi}{9}\right)\\ &= \frac{1}{4}\cos\left(\frac{4\pi}{9}\right)+\frac{1}{2}\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right)\\ &= \frac{1}{4}\cos\left(\frac{4\pi}{9}\right)+\frac{1}{4}\left(\cos\left(\frac{\pi}{9}+\frac{4\pi}{9}\right)+\cos\left(\frac{\pi}{9}-\frac{4\pi}{9}\right)\right)\\ &= \frac{1}{4}\cos\left(\frac{4\pi}{9}\right)+\frac{1}{4}\left(\cos\left(\frac{5\pi}{9}\right)+\frac{1}{2}\right)\\ &= \frac{1}{4}\cos\left(\frac{4\pi}{9}\right)+\frac{1}{4}\cos\left(\frac{5\pi}{9}\right)+\frac{1}{8}\\ &= \frac{1}{4}\cos\left(\frac{4\pi}{9}\right)-\frac{1}{4}\cos\left(\frac{4\pi}{9}\right)+\frac{1}{8}\\ &= \frac{1}{8}.\end{aligned}$$

Taking reciprocals yields $\sec\left(\frac{\pi}{9}\right)\sec\left(\frac{2\pi}{9}\right)\sec\left(\frac{4\pi}{9}\right) = 8$.

9. The polynomial $p(x) = x^3 + 20x^2 + 20x - 20$ has roots p, q, r. Determine $(p^2-1)(q^2-1)(r^2-1)$.

9. -441

Solution: Let's first write

$$\begin{aligned} p(x) &= (x-p)(x-q)(x-r) \\ &= x^3 - (p+q+r)x^2 + (pq+pr+qr) - pqr. \end{aligned}$$

Comparing to the coefficients given in the problem, it follows that $pqr = 20$, $pq + pr + qr = 20$, and $p + q + r = -20$.

So, we have Let's first do some rearranging:

$$\begin{aligned} (p^2-1)(q^2-1)(r^2-1) &= (p-1)(p+1)(q-1)(q+1)(r-1)(r+1) \\ &= (p-1)(q-1)(r-1)(p+1)(q+1)(r+1) \\ &= (pqr - (pq+pr+qr) + (p+q+r) - 1) \\ &\qquad \cdot (pqr + (pq+pr+qr) + (p+q+r) + 1) \\ &= (20 - 20 - 20 - 1)(20 + 20 - 20 + 1) \\ &= -441. \end{aligned}$$

10. Define a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}$ such that

- for any prime number p, we have $f(p) = p$;
- for all positive integers a and b, we have $f(ab) = f(a) + f(b) + 1$.

Let

$$m = \min\{f(1020), f(1021), \ldots, f(2020)\},$$
$$M = \max\{f(1020), f(1021), \ldots, f(2020)\}.$$

What is $M - m$?

10. 1992

Solution: Note that $f(n)$ is the sum of the prime factors of n plus the number of prime factors minus 1.

For the maximum, we note that 2017 is prime, so $f(2017) = 2017$. There are several ways to show that $M = 2017$. One rigorous way is as follows. For any composite number less than 2017, there are at most 11 prime factors. Since every number is at least the sum of its prime factors, we need only check that each of $f(2007)$, ..., $f(2020)$ is at most 2017.

The minimum is trickier. We will calculate, for every t, the largest n such that $f(n) \le t$. Some brute forcing yields:

t	n
1	1
2	2
3	3
4	3
5	5

We now proceed in a more systematic and iterative way. For a fixed t, we suppose that $f(n) \le t$. We have a couple of cases to consider:

- If n is prime, then $n \le t$. So, in this case, we care about when n is the largest prime that is at most t.
- If n has a prime factor p, so that $n = pq$, then we have $p + f(q) + 1 \le t$. Rewriting, we see that we need $f(q) \le t - p - 1$. So we should pick q as large as possible such that $f(q) \le t - p - 1$.

Using the above, we can continue computing the table in a very mechanical fashion (i.e. check the largest prime at most t, and check every possible prime factor of n. Some additional reasoning can show that the only possible prime factors of n are 2, 3, and 5):

t	n
1	1
2	2
3	3
4	3
5	5

t	n
6	6
7	9
8	10
9	15
10	18

t	n
11	27
12	30
13	45
14	54
15	81

t	n
16	90
17	135
18	162
19	243
20	270

t	n
21	405
22	486
23	729
24	810
25	1215

Now, we know that the largest t for which each $f(n) < t$ for all $n \in \{1020, \ldots, 2020\}$ is $t = 24$. It follows that $m = 25$.

Finally, the answer we return is $M - m = 2017 - 25 = 1992$.